Calculus 120

Curriculum

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Curriculum Overview for Grades 10-12 Mathematics

BACKGROUND AND RATIONALE

Mathematics curriculum is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society.

It is essential the mathematics curriculum reflects current research in mathematics instruction. To achieve this goal, *The Common Curriculum Framework for Grades 10–12 Mathematics: Western and Northern Canadian Protocol* has been adopted as the basis for a revised mathematics curriculum in New Brunswick. The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators and others.

The framework identifies beliefs about mathematics, general and specific student outcomes, and achievement indicators agreed upon by the seven jurisdictions. This document is based on both national and international research by the WNCP and the NCTM.

There is an emphasis in the New Brunswick curriculum on particular key concepts at each Grade which will result in greater depth of understanding and ultimately stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early Grades to ensure students develop a solid foundation in numeracy.

The intent of this document is to clearly communicate high expectations for students in mathematics education to all education partners. Because of the emphasis placed on key concepts at each Grade level, time needs to be taken to ensure mastery of these concepts. *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge (NCTM Principles and Standards, 2000).*

BELIEFS ABOUT STUDENTS AND MATHEMATICS LEARNING

The New Brunswick Mathematics Curriculum is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice. These beliefs include:

- mathematics learning is an active and constructive process;
- learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates;
- learning is most likely to occur when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking and that nurtures positive attitudes and sustained effort; and
- learning is most effective when standards of expectation are made clear with on-going assessment and feedback.

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and aspirations.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best developed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding students benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial and symbolic representations of mathematics. The learning environment should value, respect and address all students' experiences and ways of thinking, so that students are comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals for Mathematically Literate Students

The main goals of mathematics education are to prepare students to:

- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will:

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through:

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history.

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must be taught to set achievable goals and assess themselves as they work toward these goals.

Striving toward success, and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Diverse Cultural Perspectives

Students come from a diversity of cultures, have a diversity of experiences and attend schools in a variety of settings including urban, rural and isolated communities. To address the diversity of knowledge, cultures, communication styles, skills, attitudes, experiences and learning styles of students, a variety of teaching and assessment strategies are required in the classroom. These strategies must go beyond the incidental inclusion of topics and objects unique to a particular culture.

For many First Nations students, studies have shown a more holistic worldview of the environment in which they live (Banks and Banks 1993). This means that students look for connections and learn best when mathematics is contextualized and not taught as discrete components. Traditionally in Indigenous culture, learning takes place through active participation and little emphasis is placed on the written word. Oral communication along with practical applications and experiences are important to student learning and understanding. It is important that teachers understand and respond to both verbal and non-verbal cues to optimize student learning and mathematical understandings.

Instructional strategies appropriate for a given cultural or other group may not apply to all students from that group, and may apply to students beyond that group. Teaching for diversity will support higher achievement in mathematics for all students.

Adapting to the Needs of All Learners

Teachers must adapt instruction to accommodate differences in student development as they enter school and as they progress, but they must also avoid gender and cultural biases. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom. The reality of individual student differences must not be ignored when making instructional decisions.

As well, teachers must understand and design instruction to accommodate differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Designing classroom activities to support a variety of learning styles must also be reflected in assessment strategies.

Universal Design for Learning

The New Brunswick Department of Education and Early Childhood Development's definition of inclusion states that every child has the right to expect that his or her learning outcomes, instruction, assessment, interventions, accommodations, modifications, supports, adaptations, additional resources and learning environment will be designed to respect his or her learning style, needs and strengths.

Universal Design for Learning is a "…framework for guiding educational practice that provides flexibility in the ways information is presented, in the ways students respond or demonstrate knowledge and skills, and in the ways students are engaged." It also "...reduces barriers in instruction, provides appropriate accommodations, supports, and challenges, and maintains high achievement expectations for all students, including students with disabilities and students who are limited English proficient" (CAST, 2011).

In an effort to build on the established practice of differentiation in education, the Department of Education and Early Childhood Development supports *Universal Design for Learning* for all students. New Brunswick curricula are created with universal design for learning principles in mind. Outcomes are written so that students may access and represent their learning in a variety of ways, through a variety of modes. Three tenets of universal design inform the design of this curriculum. Teachers are encouraged to follow these principles as they plan and evaluate learning experiences for their students:

- **Multiple means of representation:** provide diverse learners options for acquiring information and knowledge
- **Multiple means of action and expression:** provide learners options for demonstrating what they know
- **Multiple means of engagement:** tap into learners' interests, offer appropriate challenges, and increase motivation

For further information on *Universal Design for Learning*, view online information at http://www.cast.org/.

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in literacy, science, social studies, music, art, and physical education.

NATURE OF MATHEMATICS

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this document. These components include: **change**, **constancy**, **number sense**, **patterns**, **relationships, spatial sense** and **uncertainty**.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as:

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

Students need to learn that new concepts of mathematics as well as changes to already learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS–Benchmarks, 1993, p. 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include:

- the area of a rectangular region is the same regardless of the methods used to determine the solution
- the sum of the interior angles of any triangle is 180°
- the theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

Many important properties in mathematics do not change when conditions change. Examples of constancy include:

- the conservation of equality in solving equations
- the sum of the interior angles of any triangle
- the theoretical probability of an event.

Number Sense

Number sense, which can be thought of as deep understanding and flexibility with numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p. 146). Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. Evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing mathematically rich tasks that allow students to make connections.

Patterns

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships. Within the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

ASSESSMENT

Ongoing, interactive assessment (*formative assessment*) is essential to effective teaching and learning. Research has shown that formative assessment practices produce significant and often substantial learning gains, close achievement gaps and build students' ability to learn new skills (Black & William, 1998, OECD, 2006). Student involvement in assessment promotes learning. Interactive assessment, and encouraging self-assessment, allows students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes:

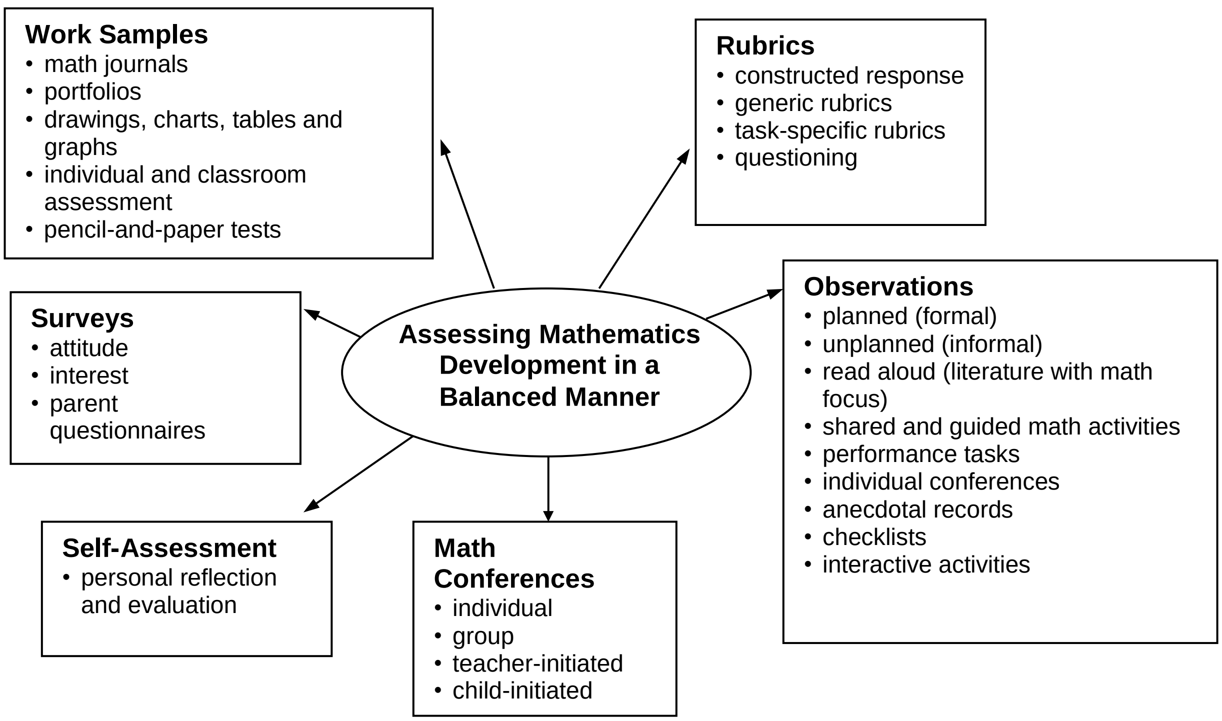
- providing clear goals, targets and learning outcomes
- using exemplars, rubrics and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning (Davies, 2000)

Formative assessment practices act as the scaffolding for learning which, only then, can be measured through summative assessment. *Summative assessment,* or assessment *of* learning, tracks student progress, informs instructional programming and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning and produce achievement gains.

Student assessment should:

- align with curriculum outcomes
- use clear and helpful criteria
- promote student involvement in learning mathematics during and after the assessment experience
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction

(adapted from: NCTM, Mathematics Assessment: A practical handbook, 2001, p.22)



CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.

GRADE TOPICS	10 11 12
The topics of study vary between pathways for Grades 10–12 mathematics. Topics include: • *Algebra* • *Financial Mathematics* • *Geometry* • *Logical Reasoning* • *Mathematics Research Project* • *Measurement* • *Number* • *Permutations, Combinations and Binomial Theorem* • *Probability* • *Relations and Functions* • *Statistics* • *Trigonometry* • *Differential and Integral Calculus*	**GENERAL OUTCOMES** **SPECIFIC OUTCOMES** **ACHIEVEMENT INDICATORS**

NATURE OF MATHEMATICS

Change
Constancy
Number Sense
Patterns
Relationships
Spatial Sense
Uncertainty

MATHEMATICAL PROCESSES:
Communication, Connections, Mental Mathematics and Estimation, Problem Solving, Reasoning, Technology, Visualization

MATHEMATICAL PROCESSES

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics. Students are expected to:

- communicate in order to learn and express their understanding of mathematics (Communications: C)
- develop and apply new mathematical knowledge through problem solving (Problem Solving: PS)
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines (Connections: CN)
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation: ME)
- select and use technologies as tools for learning and solving problems (Technology: T)
- develop visualization skills to assist in processing information, making connections and solving problems (Visualization: V).
- develop mathematical reasoning (Reasoning: R)

The New Brunswick Curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, knowledge, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication can help students make connections among concrete, pictorial, symbolic, verbal, written and mental representations of mathematical ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all Grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts. Problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics.

When students encounter new situations and respond to questions of the type, *How would you...?* or *How could you ...?,* the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learnings in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families or current events.

Both conceptual understanding and student engagement are fundamental in moulding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive and deductive reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly look for, and engage in, finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students can begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences, and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"*Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from which learners extract understanding… Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*" (Caine and Caine, 1991, p. 5).

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

"Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental mathematics" (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "*become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving*" (Rubenstein, 2001).

Mental mathematics "*provides a cornerstone for all estimation processes offering a variety of alternate algorithms and non-standard techniques for finding answers*" (Hope, 1988).

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values.

Students need to know how, when and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Technology [T]

Technology can be used effectively to contribute to and support the learning of a wide range of mathematical outcomes. Technology enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

Calculators and computers can be used to:
- explore and demonstrate mathematical relationships and patterns
- organize and display data
- generate and test inductive conjectures
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- model situations
- develop number and spatial sense.

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all Grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization “involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world” (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure and when to estimate and involves knowledge of several estimation strategies (Shaw and Cliatt, 1989, p. 150).

Visualization is fostered through the use of concrete materials, technology and a variety of visual representations. It is through visualization that abstract concepts can be understood concretely by the student. Visualization is a foundation to the development of abstract understanding, confidence and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking.

Questions that challenge students to think, analyze and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, *Why do you believe that's true/correct?* or *What would happen if ….*

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

ESSENTIAL GRADUATION LEARNINGS

Graduates from the public schools of Atlantic Canada will be able to demonstrate knowledge, skills, and attitudes in the following essential graduation learnings. These learnings are supported through the outcomes described in this curriculum document.

Aesthetic Expression
Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.

Citizenship
Graduates will be able to assess social, cultural, economic, and environmental interdependence in a local and global context.

Communication
Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) as well as mathematical and scientific concepts and symbols to think, learn, and communicate effectively.

Personal Development
Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.

Problem Solving
Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language, mathematical, and scientific concepts.

Technological Competence
Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems

PATHWAYS AND TOPICS

The Common Curriculum Framework for Grades 10–12 Mathematics on which the New Brunswick Grades 10-12 Mathematics curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics*. In New Brunswick all Grade 10 students share a common curriculum covered in two courses: *Geometry, Measurement and Finance 10* and *Number, Relations and Functions 10*. Starting in Grade 11, three pathways are available: *Finance and Workplace*, *Foundations of Mathematics*, and *Pre-Calculus*.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. Students are encouraged to cross pathways to follow their interests and to keep their options open. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all three pathways are to provide prerequisite attitudes, knowledge, skills and understandings for specific post-secondary programs or direct entry into the work force. All three pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of each pathway has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* and on consultations with mathematics teachers.

Financial and Workplace Mathematics

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, statistics and probability.

Foundations of Mathematics

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, statistics and probability.

Pre-calculus

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Students develop of a function tool kit including quadratic, polynomial, absolute value, radical, rational, exponential, logarithmic and trigonometric functions. They also explore systems of equations and inequalities, degrees and radians, the unit circle, identities, limits, derivatives of functions and their applications, and integrals.

Outcomes and Achievement Indicators

The New Brunswick Curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes and achievement indicators.

General Curriculum Outcomes (GCO) are overarching statements about what students are expected to learn in each strand/sub-strand. The general curriculum outcome for each strand/sub-strand is the same throughout the pathway.

Specific Curriculum Outcomes (SCO) are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as required for a given Grade.

Achievement indicators are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome. In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for clarification and are not requirements that must be addressed to fully meet the learning outcome. The word *and* used in an outcome indicates that both ideas must be addressed to fully meet the learning outcome, although not necessarily at the same time or in the same question.

Instructional Focus

Each pathway in *The Common Curriculum Framework for Grades 10–12 Mathematics* is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful. Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches, and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning, assessment as learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

HIGH SCHOOL MATHEMATICS PATHWAYS

The New Brunswick K–12 mathematics curriculum prepares students for whatever they choose to do after high school. The curriculum is focused on providing students with mathematics skills and funds of knowledge to confidently solve problems and to strengthen their New Brunswick Global Competencies. This document can be used together by educators and learners.

High school math courses are organized into three pathways: **Financial and Workplace Mathematics**, **Foundations of Mathematics**, and **Pre-Calculus**. Each pathway is organized to provide specific mathematical contexts, concepts and skills; all pathways provide students with the mathematical literacy and New Brunswick Global Competencies growth they need. Students should select courses in the pathway that best fits their interests and plans for after high school.

FINANCIAL AND WORKPLACE MATHEMATICS

This pathway is organized for students who plan to take post-secondary programs that require applied mathematics or who plan to enter the workforce directly after high school.

FOUNDATIONS OF MATHEMATICS

This pathway is organized for students who plan to take post-secondary academic programs that **do not require** calculus.

PRE-CALCULUS

This pathway is organized for students who plan to take post-secondary programs that **require calculus.**

Are pathways different from levels?

Levelled courses are developed for different academic abilities. Pathways are organizations of courses that are conceptually similar and work towards specific student interests or needs after high school. Pathways are not sorted by difficulty/academic ability.

If a student changes their graduation plans, can they switch mathematics pathways?

Students may enrol in any high school mathematics course provided the specific prerequisites are completed. *Number, Relations, and Functions 10* and all other Grade 11 and Grade 12 mathematics courses, provide credits towards graduation. Credits towards graduation may be acquired from face-to-face or online courses. For example, students looking to complete *Number, Relations and Functions 10* in a later academic year may complete the online version hosted on the NBVLC.

Do I have to complete all the courses on a specific mathematics pathway to graduate?

No. Courses used for credit do not have to be on the same pathway provided the student meets prerequisite requirements. However, students may need specific courses for entrance to post-secondary programs. Please verify with a high school guidance/career counsellor and with the institution's entrance requirements.

Should students take more math courses even if they don't need them to graduate?

Students may need specific courses for entrance to post-secondary programs. Taking additional mathematics courses offers continued acquisition of mathematics skills and funds of knowledge, and strengthens mathematical literacy and New Brunswick Global Competencies. Further mathematics connections and understandings may be made by working with mathematics in different contexts.

For high school graduation, New Brunswick students will successfully complete:

9/10 Program common curriculum:
Mathematics 9 (Grade 9); and
Geometry, Measurement and Finance 10 (Grade 10).

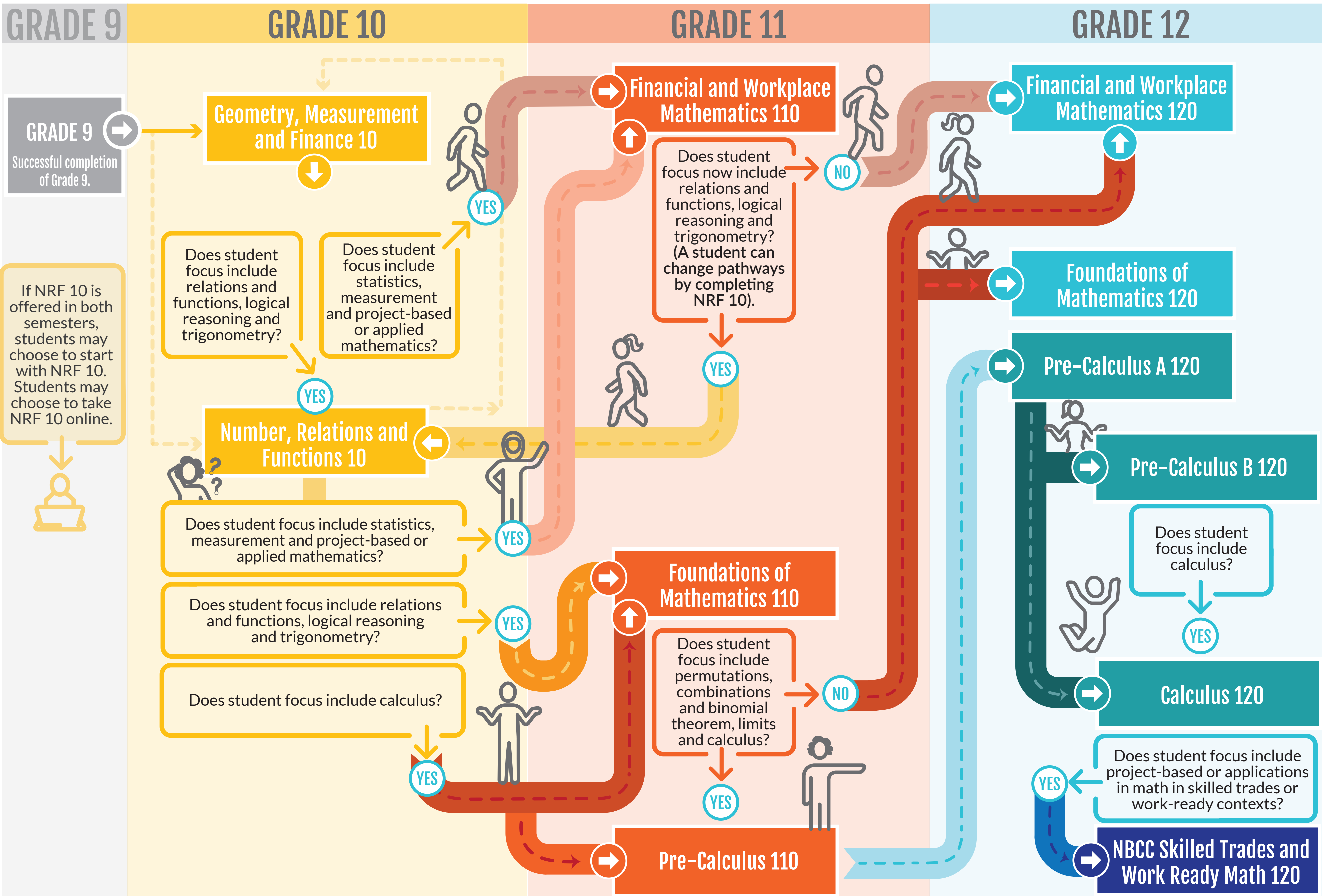
Plus acquire two additional math credits which can include:

Number, Relations and Functions 10 (Grade 10);
Financial and Workplace Mathematics 110; and
Foundations of Mathematics 110.

MyBluePrint:

Anglophone East
Anglophone North
Anglophone West
Anglophone South

HIGH SCHOOL MATHEMATICS PATHWAYS



HIGH SCHOOL MATHEMATICS PATHWAYS

GRADE 10

Geometry, Measurement and Finance 10
EN/FR

Length: One semester

Prerequisite: *Mathematics 9*

Topics: Pythagorean Theorem; polygons; angles; trigonometric ratios; metric and imperial systems of measurement; surface area and volume; unit pricing; currency exchange; income (gross and net pay); credit cards; loans; and interest

Number, Relations and Functions 10
EN/FR

Length: One semester

Prerequisite: *Mathematics 9*

Topics: prime factors; common factors; square and cube roots; irrational numbers; integral and rational exponents; polynomial expressions; trinomial factoring; linear relations and functions; slope; distance formula; and midpoint formula

FINANCIAL AND WORKPLACE MATHEMATICS

This pathway is organized for students who plan to take post-secondary programs that require applied mathematics or who plan to enter the workforce directly after high school.

FOUNDATIONS OF MATHEMATICS

This pathway is organized for students who plan to take post-secondary academic programs that do not require calculus.

PRE-CALCULUS

This pathway is organized for students who plan to take post-secondary programs that require calculus.

GRADE 11

Financial and Workplace Mathematics 110
EN/FR

Length: One semester

Prerequisite: *Geometry, Measurement, and Finance 10*

Topics: right triangles; trigonometry; scale models and drawings; numerical reasoning; renting and buying; investment portfolios; personal budgets; application of formulas; slope; and proportional reasoning

Foundations of Mathematics 110
EN/FR

Length: One semester

Prerequisites: *Number, Relations, and Functions 10* **and** *Geometry, Measurement, and Finance 10*

Topics: numerical and logical reasoning; angles and triangles; sine and cosine laws; systems of linear inequalities; quadratic functions; renting and buying; and investment portfolios

Pre-Calculus 110
EN/FR

Length: One semester

Pre or Co-requisite: *Foundations of Mathematics 110*

Topics: absolute value functions; radical expressions and equations; rational expressions and equations; angles and trigonometric ratios (0°–360°); polynomial factoring; systems of equations; quadratic functions and equations; and linear and quadratic inequalities

GRADE 12

Financial and Workplace Mathematics 120

Length: One semester

Prerequisites: *Financial and Workplace Mathematics 110* **or** *Foundations of Mathematics 110*

Topics: measuring; sine and cosine laws; properties of polygons; transformations of 2-D and 3-D shapes; small business finance; linear relationships; data interpretation; and probability

Foundations of Mathematics 120

Length: One semester

Prerequisite: *Foundations of Mathematics 110*

Topics: normal distribution; standard deviation; confidence intervals; set theory; conditional statements; probability; binomial theorem; and polynomial, exponential, logarithmic and sinusoidal functions

Pre-Calculus A 120
EN/FR

Length: One semester

Prerequisite: *Pre-Calculus 110*

Topics: graphs of functions and related equations; inverse, radical, exponential and logarithmic functions; angles in standard position in degrees and radians; unit circle; trigonometric ratios and sine, cosine and tangent equations to solve problems; and trigonometric identities

Note: Most post-secondary programs that require Pre-Calculus A 120 **also** require Pre-Calculus B 120

NBCC Skilled Trades and Work Ready 120

Length: One semester

Prerequisite: *Geometry, Measurement, and Finance 10*

Topics (in a skilled trade and/or work-ready context): whole numbers; fractions; decimals; percent; ratio and proportion, integers; scientific notation; metric system; and measurement

Note: This course is intended for learners entering an NBCC program in the next academic year. Any remaining seats may be filled at the school's discretion.

Pre-Calculus B 120

Length: One semester

Pre or Co-requisite: *Pre-Calculus A 120*

Topics: arithmetic and geometric sequences and series; polynomial factoring and functions; reciprocal and rational functions; function toolkit permutations; combinations and binomial theorem; and limits and continuity of functions

Calculus 120

Length: One semester

Prerequisites: *Pre-Calculus A 120* **and** *Pre-Calculus B 120*

Topics: graphs of functions and related equations; inverse, radical, exponential and logarithmic functions; angles in standard position in degrees and radians; unit circle; trigonometric ratios and sine, cosine and tangent equations to solve problems; and trigonometric identities

EN/FR = ENGLISH/FRENCH

Available online

CURRICULUM DOCUMENT FORMAT

This guide presents the mathematics curriculum by Grade level so that a teacher may readily view the scope of the outcomes which students are expected to meet during that year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings at a particular Grade level are part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes (GCOs).

The heading of each page gives the General Curriculum Outcome (GCO), and Specific Curriculum Outcome (SCO). The key for the mathematical processes follows. A Scope and Sequence is then provided which relates the SCO to previous and next Grade SCO's. For each SCO, Elaboration, Achievement Indicators, Suggested Instructional Strategies, and Suggested Activities for Instruction and Assessment are provided. For each section, the *Guiding Questions* should be considered.

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Mathematical Processes

[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Math **[T]** Technology **[V]** Visualization **[R]** Reasoning and Estimation

Scope and Sequence

Previous Grade or Course SCO's	Current Grade SCO	Following Grade or Course SCO's

Elaboration

Describes the "big ideas" to be learned and how they relate to work in previous Grades

Guiding Questions:
- *What do I want my students to learn?*
- *What do I want my students to understand and be able to do?*

Achievement Indicators

Describes observable indicators of whether students have met the specific outcome

Guiding Questions:
- *What evidence will I look for to know that learning has occurred?*
- *What should students demonstrate to show their understanding of the mathematical concepts and skills?*

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Suggested Instructional Strategies

General approach and strategies suggested for teaching this outcome

Guiding Questions
- *What learning opportunities and experiences should I provide to promote learning of the outcomes and permit students to demonstrate their learning?*
- *What teaching strategies and resources should I use?*
- *How will I meet the diverse learning needs of my students?*

Suggested Activities for Instruction and Assessment

Some suggestions of specific activities and questions that can be used for both instruction and assessment.

Guiding Questions
- *What are the most appropriate methods and activities for assessing student learning?*
- *How will I align my assessment strategies with my teaching strategies?*

Guiding Questions
- *What conclusions can be made from assessment information?*
- *How effective have instructional approaches been?*
- *What are the next steps in instruction?*

Calculus 120

Specific Curriculum Outcomes

SCO: **C1: Explore the concepts of average and instantaneous rate of change.** [CN, ME, R, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

Calculus

C1. Explore the concepts of average and instantaneous rate of change.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
RF3, RF4, RF5, RF6, RF7 L1, L2, L3	**C1.** Explore the concepts of average and instantaneous rate of change.

ELABORATION

In Grade 10, students explored the concept of slope and equations of a line. They calculated the slope of a line by choosing any two points on that line and dividing the difference in the y-values by the difference in the x-values. They learned that any two points can be used to determine the slope of a line because the slope is constant, that the equation of a linear function can be derived given the coordinates of a point on the line and the slope of the line, and that the slopes of parallel lines are the same, and the slopes of perpendicular lines are negative reciprocals of each other.

In this outcome students will extend the concept of the slope to explore the **average rate of change** of a quantity over a period of time as the amount of change divided by the elapsed time. As the amount of elapsed time decreases, the average rate of change approaches the **instantaneous rate of change** at a particular instant. This concept can be generalized for working with functions.

The slope of the secant line to the curve $y = f(x)$ joining the points $(x, f(x))$ and $(x + h, f(x + h))$ on the curve, is:

$$m_{secant} = \frac{f(x+h) - f(x)}{h}$$

If the distance between the two points on the curve, h, decreases, then this slope approaches the value of the slope of the curve at the point $(x, f(x))$. We can determine this slope by finding the limit of the above expression as $h \to 0$:

$$m_{curve} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The line that passes through this point and has the same slope as the curve is called the **tangent line** to the curve at that point. The line that passes through this point and is perpendicular to the tangent line is called the **normal line** to the curve at that point. Students will determine the equation of a tangent and normal line to a curve at a given point.

Students will explore a variety of applications of average and instantaneous rates of change as the **average speed**, the distance travelled by an object divided by the elapsed time, and as the **instantaneous speed**, the speed of an object at a given instant in time.

The average rates of change introduced in this section provide an application related to limits of rational functions. They are also related to derivatives, although the word *derivative* is not used until a later outcome.

SCO: **C1: Explore the concepts of average and instantaneous rate of change.** [CN, ME, R, V]

ACHIEVEMENT INDICATORS

- Determine the average rate of change (slope of a secant line) of a function over an interval.
- Demonstrate an understanding that the instantaneous rate of change of a function at a point (slope of a tangent line) is the limiting value of a sequence of average rates of change.
- Determine the slope of a curve at a given point.
- Determine whether a curve has a tangent line at a given point.
- Determine the equation of a tangent line to a curve at a given point.
- Determine the equation of a normal line to a curve at a given point.
- Calculate and interpret average rates of change drawn from a variety of applications.
- Solve problems involving instantaneous rates of change drawn from a variety of applications.

Suggested Instructional Strategies

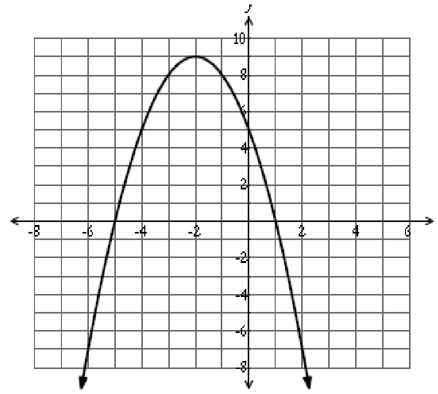
- Provide students with the graph of a nonlinear function such as $f(x) = x^3 - 2x^2 - 5x + 1$ and ask them to describe where the slope of the function is positive, zero and negative. Students should observe that there is no single slope for this graph since the function increases (slope is positive), levels off (slope is zero), and decreases (slope is negative) for various intervals of x.
- Before students determine the slope of a function at a particular point (i.e., the slope of the tangent line to the curve at a particular point), they must first be able to distinguish between a tangent line and a secant line. Using graphing technology, show the graph of the polynomial $f(x) = x^3 - 2x^2 - 5x + 1$ on the same grid as the graphs of $y = 2x + 5$, $y = -x - 7$ and $y = \frac{1}{2}x + 2$. Ask students to identify which of these lines appear to touch the curve at one point.
- The link http://www.geogebratube.org/student/m16248 can be used to emphasize the concept of average and instantaneous rates of change. This applet shows the average rate of change of a function as a secant line. To visualize the instantaneous rate of change move the intersection points to make the secant line approach the tangent line.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Given the graph below, determine the average rate of change over the following intervals.

a) $x = -4$ to $x = 1$
b) $x = -5$ to $x = -1$
c) $x = -6$ to $x = -3$

$Answers\text{:}\ \ a)\ m = -1\quad b)\ m = 2\quad c)\ m = 5$



Q Find the average rate of change of each function over the indicated interval.

a) $y = 4x - 3x^2,\ \ [2,3]$ $\quad Answers\text{:}\ a)\ (2,-4)(3,-15)\ m = -11$
b) $y = \sqrt{x},\ \ [0,2]$ $\quad b)\ (0,0)\ (2,\sqrt{2})\ m = \frac{\sqrt{2}}{2}$
c) $y = \frac{2x+1}{x+2}\ ,\ \ [1,3]$ $\quad c)\ (1,1)\left(3,\frac{7}{5}\right)\ m = \frac{1}{5}$

Q An object dropped from rest from the top of a cliff falls $h(t) = -4.9t^2 + 50$ metres in the first t seconds.

a) Find the average speed during the first 3 seconds of fall. $(Answer\text{:}\ -14.7\ m/s)$
b) Find the speed of the object 3 seconds after it has been dropped. $(Answer\text{:}\ -29.4\ m/s)$

Q For each function at the indicated x-value, find

- the slope of the curve;
- the equation of the tangent line;
- the equation of the normal line.

a) $y = 4x - x^2,\ \ x = 1$ $\quad Answer\text{:}\ \ a)\ m_{tan} = 2\ \ tan\ line\ y = 2x + 1,\ norm\ line\ y = -\frac{1}{2}x + \frac{7}{2}$
b) $y = x - x^3,\ \ x = -1$ $\quad b)\ m_{tan} = -2\ \ tan\ line\ \ y = -2x - 2,\ norm\ line\ \ y = \frac{1}{2}x + \frac{1}{2}$
c) $y = \frac{2}{x+1}\ ,\ \ x = -2$ $\quad c)\ m_{tan} = -2\ \ tan\,line\ \ y = -2x - 6,\ norm\ line\ \ y = \frac{1}{2}x - 1$

Q Find the instantaneous rate of change of the position function $y = t^2 - 8t$ in metres, at $t = 2$ seconds. $Answer\text{:}\ \ m = -4\ m/s$

Q What is the rate of change of the area of a square with respect to the length of a side when the length is $5\ cm$? $Answer\text{:}\ m = 10\ cm^2/cm$

Q Explain why you cannot find the equation of a tangent line to the curve $y = \frac{x+1}{x-2}$ at $x = 2$.
$Answer\text{:}\ x - 2 \neq 0,\ \ \therefore x \neq 2\ \ There\ is\ a\ vertical\ asymptote\ at\ x = 2$

SCO: **C2:** Determine the derivative of a function by applying the definition of derivative. [CN, ME, R, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C2. Determine the derivative of a function by applying the definition of derivative.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
RF3, RF4, RF5, RF6, RF7 L1, L2, L3	**C2.** Determine the derivative of a function by applying the definition of a derivative.

ELABORATION

In *Pre-Calculus B 120*, students graphed and analyzed polynomial functions, limited to polynomial functions of degree less than or equal to 5.

For this outcome, students will develop and use the definition of a derivative in which the slope of a curve $y = f(x)$ at the point $(a, f(a))$ is generalized to any point on the curve $(x, f(x))$. **This limit process is the central concept of calculus**. If a limit exists, the derivative of the function f, with respect to the variable x, is the function f' whose value at x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The different notations for the derivative should be discussed. Derivatives will be limited to polynomials of degree 3 or less, to square root functions, and to rational functions with linear terms.

Students will learn to sketch derivative graphs given an original function and conversely sketch the original function, given a derivative graph, a skill often stressed in post-secondary calculus courses. The focus at this point should be to draw rough sketches. Sketching graphs in more detail will be covered later in this course.

To sketch graphs students should be aware that:

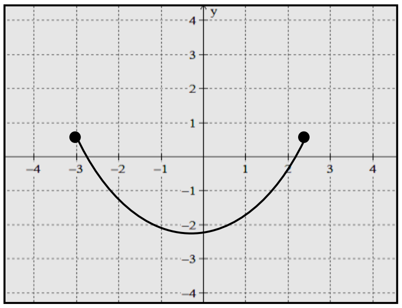
- When the graph of f has a positive slope, the graph of f' will be above the x-axis.
- When the graph of f has zero slope, the graph of f' will be on the x-axis.
- When the graph of f has a negative slope, the graph of f' will be below the x-axis.

Students will also explore the differentiability of functions. A function f is said to be **differentiable** at a if $f'(a)$ exists. A function f is said to be **differentiable on an interval** if it is differentiable at every number in the interval. A **right-hand derivative** is a derivative defined by a right-hand limit. A **left-hand derivative** is a derivative defined by a left-hand limit.

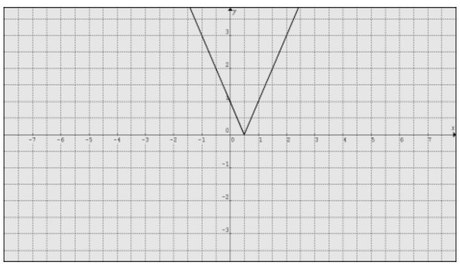
SCO: **C2:** Determine the derivative of a function by applying the definition of derivative. [CN, ME, R, V]

There are five situations in which relations are <u>not</u> differentiable, as illustrated below:

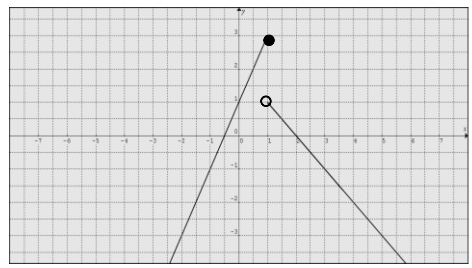
1) Function with a restricted domain *(not differentiable at the endpoints)*



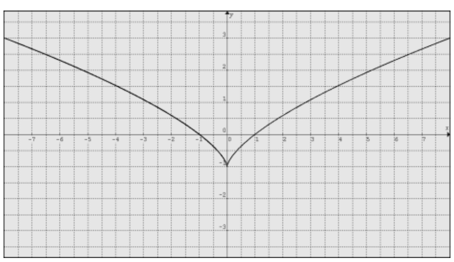
2) Corner



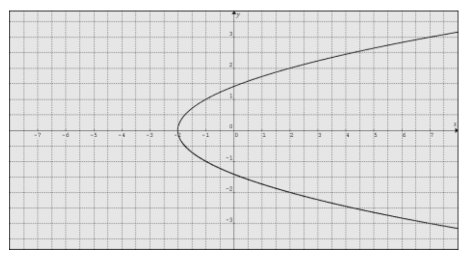
3) Discontinuity



4) Cusp



5) Vertical Tangent



ACHIEVEMENT INDICATORS

- Determine the derivative of a function, $f(x)$ by using the limit definition of the derivative,
  $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{or} \quad f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$
- Define and evaluate the derivative at $x = a$.
- Use alternate notation interchangeably to express derivatives (*i.e.,* $f'(x)$, $\frac{dy}{dx}$, y', $etc.$).
- Given the graph of the derivative of a function, sketch a graph of the function.
- Given the graph of a function, sketch a graph of its derivative.
- Determine whether a function is differentiable at a given point.
- Explain why a function is not differentiable at a given point, and distinguish among corners, cusps, discontinuities, and vertical tangents.
- Determine all values for which a function is differentiable.

SCO: **C2:** Determine the derivative of a function by applying the definition of derivative. [CN, ME, R, V]

Suggested Instructional Strategies

- This section can be introduced by showing the graph of a function and how the slopes of secant lines approach a limit corresponding to the slope of a tangent line at a given point.
- Calculating derivatives can be learned by using the definition by first, calculating the derivative at a particular value $x = a$, and then, generalizing to find the derivative at a general point x.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Find the value of the derivative of each of the following functions using the definition of derivative at the indicated value of a.

a) $f(x) = 5x - 9x^2, a = 1$ *Answers:* a) $f'(1) = -13$

b) $f(x) = \frac{1}{\sqrt{x}}, a = 4$ b) $f'(4) = -\frac{1}{16}$

c) $f(x) = \frac{x^2-1}{2x-3}, a = 2$ c) $f'(2) = -2$

Q Differentiate each of the following functions using the definition of derivative.

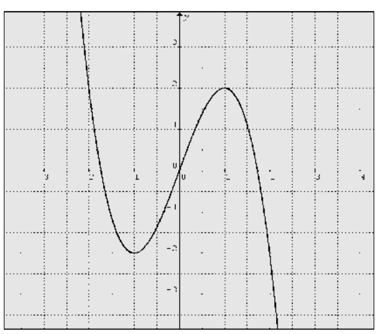
a) $f(x) = x^2 - 2x^3$ *Answers:* a) $f'(x) = 2x - 6x^2$

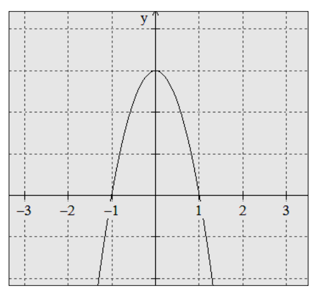
b) $f(x) = \sqrt{9-x}$ b) $f'(x) = \frac{-1}{2\sqrt{9-x}}$

c) $f(x) = \frac{1-2x}{3+x}$ c) $f'(x) = \frac{-7}{(3+x)^2}$

Q Sketch the graph of the derivative of the function whose graph is shown below.

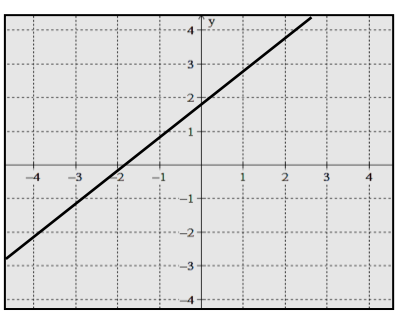
Possible Answer:

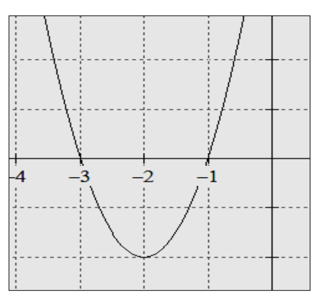




Q Sketch a possible graph of $f(x)$, given the graph of $f'(x)$.

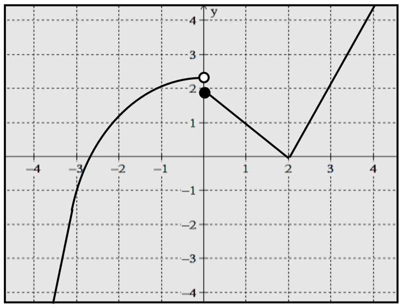
Possible Answer:





SCO: **C2:** Determine the derivative of a function by applying the definition of derivative. [CN, ME, R, V]

Q At what point(s) is (are) the graph of the following function not differentiable? Why?



Answer: At $x = 0$, the function is not differentiable because there is a jump in the graph (discontinuous). At $x = 2$, the function is not differentiable because it is a corner function

Q Each of the following functions fails to be differentiable at $x = 1$. Determine whether the problem is a corner, cusp, vertical tangent, or discontinuity.

a) $y = (x-1)^{2/3}$

b) $y = |x-1|$

c) $y = \frac{1}{x-1}$

d) $y = \sqrt{x-1}$

Answers: a) cusp b) corner c) discontinuity d) vertical tangent

Q Determine all values of x for which each function is not differentiable.

a) $f(x) = \frac{x}{x-5}$

b) $f(x) = \sqrt[3]{2x-6}$

c) $f(x) = \sin|x+\pi|$

d) $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$

Answers: a) $x = 5$ b) $x = 3$ c) $x = k\pi, k\epsilon I$ d) $x = 0$

SCO: **C3: Apply derivative rules to determine the derivative of a function, including: Constant Rule; Power Rule; Constant Multiple Rule; Sum Rule; Difference Rule; Product Rule; Quotient Rule.** [C, CN, PS, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

C3: Apply derivative rules to determine the derivative of a function, including: Constant Rule; Power Rule; Constant Multiple Rule; Sum Rule; Difference Rule; Product Rule; Quotient Rule.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
L1, L2, L3	**C3** Apply derivative rules to determine the derivative of a function, including: Constant Rule; Power Rule; Constant Multiple Rule; Sum Rule; Difference Rule; Product Rule; Quotient Rule.

ELABORATION

For this outcome students will apply the derivative rules listed below. Teachers should derive the **Constant, Sum, Difference, Product** and **Quotient Rules** with students to build conceptual understanding. However, students are not responsible for these proofs for assessment.

RULES FOR DIFFERENTIATION		
Constant Rule	If f is a function with the constant value c then:	$\frac{d}{dx}(c) = 0$
Constant Multiple Rule	If u is a differentiable function of x and c is a constant then:	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
Power Rule	If n is a positive or negative integer and $x \neq o$ then:	$\frac{d}{dx}(x^n) = nx^{n-1}$
Difference Rule	If u and v are differentiable functions of s, then their sum and difference are differentials at every point where u and v are differentiable. At such points:	$\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$
Sum Rule		$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$
Product Rule	If the product of two differentiable functions u and v is differentiable then:	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient Rule	At a point where $v \neq 0$, the quotient $y = u/v$ of two differentiable functions is differentiable then:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Teachers should use a graph and the definition of the derivative when introducing the **Constant Rule**. When graphing a horizontal line, such as $f(x) = 3$, at any point, the tangent line is the same as the horizontal line which has a slope of zero. It then follows that the derivative at any point must also be zero. Applying the definition of the derivative verifies that if $f(x) = c$ where c is a constant, the $f'(x) = 0$ or $\frac{d}{dx}(c) = 0$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

The **Constant Multiple Rule** states that the derivative of a constant times a function is equivalent to the constant times the derivative of the function. To verify, the definition of the derivative can be applied:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = cf'(x)$$

In order to bring the constant c outside the limit, the limit law is applied, which states that the limit of a constant times a function is the constant times the limit of the function.

SCO: **C3: Apply derivative rules to determine the derivative of a function, including: Constant Rule; Power Rule; Constant Multiple Rule; Sum Rule; Difference Rule; Product Rule; Quotient Rule.** [C, CN, PS, R]

The **Power Rule** states that to differentiate x^n, multiply by n and subtract 1 from the exponent. For example, the derivative of x^3 is $3x^2$ and the derivative of $4x^2$ is $8x$.

The **Difference Rule** can be proven using the definition of the derivative:

$$k'(x) = \lim_{h\to 0}\frac{k(x+h)-k(x)}{h} = \lim_{h\to 0}\frac{f(x+h)-g(x+h)-((f(x)-g(x))}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)-g'(x)$$

This can also be written as $\frac{d}{dx}(f-g) = \frac{d}{dx}(f) - \frac{d}{dx}(g)$. Similarly the **Sum Rule** states that the derivative of a sum of two functions is equal to the sum of the derivatives of each function.

The **Product Rule** states that if $f(x)$ and $g(x)$ are differentiable then the product $k(x) = f(x)g(x)$ is differentiable. The definition of the derivative can be used to prove that $k'(x) = f'(x)g(x) + f(x)g'(x)$.

Similarly, the **Quotient Rule** states that if two functions $f(x)$ and $g(x)$ are differentiable, then the quotient is differentiable. The **Quotient Rule** can be derived using the definition of a derivative, or by using the **Product Rule**. When using these rules, students do not need to simplify unless instructed.

The differentiation process should be continued to find the second, third, and successive derivatives of $f(x)$, which are called higher order derivatives of $f(x)$.

The **second derivative** is found by taking the derivative of the first derivative and is written as $y''=f''(x) = \frac{d^2y}{dx^2}$. The **third derivative** would be the derivative of the second derivative and is written as $y''' = f'''(x) = \frac{d^3y}{dx^3}$. The multiple prime notation will eventually become cumbersome for higher derivatives. The notation for the n^{th} derivative of $f(x)$ can be indicated as $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$.

It should be clear to students that the superscripts used for higher-order derivatives, such as $y^{(4)}$for the fourth derivative of y, are not exponents. The presence of the parenthesis in the exponent denotes differentiation, for example, $f^{(2)}(x) = f''(x)$, while the absence of the parenthesis denotes exponents for example, $f^2(x) = [f(x)]^2$.

Derivatives have a variety of applications. As discussed earlier in the course, the instantaneous rate of change of f with respect to x at a is the derivative:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

An object moving along a coordinate line has a position, s, as a function of time, t. Since **velocity** is the rate of change of the position of an object with respect to time, the velocity, $v(t)$, will be the derivative of the position function with respect to time, or $v(t) = \frac{ds}{dt}$. **Speed** is the magnitude of the velocity $v(t)$, or $speed = |v(t)| = \left|\frac{ds}{dt}\right|$.

The object's **displacement** is the difference between the path of the initial and final position covered by a moving object, calculated as: $\Delta s = f(t+\Delta t) - f(t)$.

SCO: **C3: Apply derivative rules to determine the derivative of a function, including: Constant Rule; Power Rule; Constant Multiple Rule; Sum Rule; Difference Rule; Product Rule; Quotient Rule.** [C, CN, PS, R]

Average velocity is the displacement divided by time travelled, calculated as:

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

And **instantaneous velocity** is the derivative of a position function with respect to time, calculated as:

$$v(t) = \lim_{t \to 0} \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

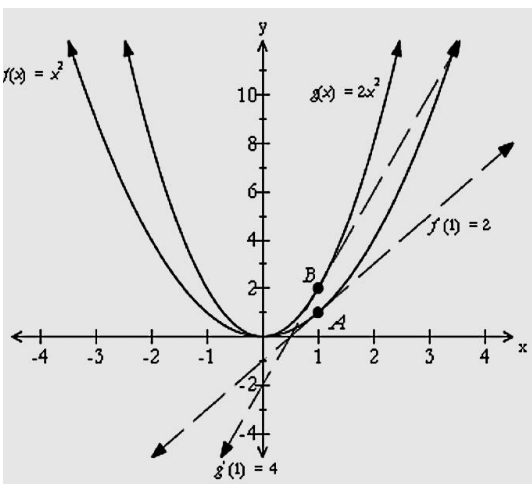
Since **acceleration** is the rate of change of the velocity of an object with respect to time, the acceleration, or $a(t)$, will be the derivative of the velocity function with respect to time, or $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

ACHIEVEMENT INDICATORS

- Determine the derivatives of functions, using the Constant, Power, Constant Multiple, Sum, Difference, Product, and Quotient Rules.
- Determine second and higher-order derivatives of functions.
- Solve problems involving derivatives drawn from a variety of applications (including applications in the social and natural sciences)

Suggested Instructional Strategies

- The **Constant Multiple Rule** can be illustrated graphically by showing that if a function $g(x)$ is created by multiplying a function, $f(x)$ by a constant c, this new function is displaced vertically by the value of that constant. For example, the graph below illustrates that if $c = 2$, the curve $g(x)$ is displaced vertically from $f(x)$ by a factor of two and the slope of the tangent line to $g(x)$ is twice that of the tangent line to $f(x)$.



- It is important to emphasize with students that the **Power Rule** applies only to power functions such as $y = x^n$. It does not apply to exponential functions such as $y = 2^x$. Students will study the derivatives of exponential functions later in this course.

SCO: **C3: Apply derivative rules to determine the derivative of a function, including: Constant Rule; Power Rule; Constant Multiple Rule; Sum Rule; Difference Rule; Product Rule; Quotient Rule.** [C, CN, PS, R]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Find the derivative of each of the following functions.

a) $f(x) = \frac{1}{2}x^6 - 3x^4 + x$

b) $f(x) = (1 + x + x^2)(2 - x^4)$

c) $f(x) = (x^3 - 2x)(x^{-4} + x^{-2})$

d) $f(x) = \frac{x+1}{x^2+x-2}$

Answers: a) $f'(x) = 3x^5 - 12x^3 + 1$

b) $f'(x) = (1 + 2x)(2 - x^4) + (-4x^3)(1 + x + x^2)$

c) $f'(x) = (3x^2 - 2)(x^{-4} + x^{-2}) + (-4x^{-5} - 2x^{-3})(x^3 - 2x)$

d) $f'(x) = \frac{(x^2+x-2)-(2x+1)(x+1)}{(x^2+x-2)^2}$

Q For each of the following, find an equation for the line tangent to the curve at the indicated x-value.

a) $y = (1 + 2x)^2$, $x = 1$

Answers: a) $when\ x = 1, y = 9\ \ b = -3\ \ y' = m = 4(1 + 2x)\ \ \therefore y = 12x - 3$

b) $y = \frac{3x+1}{x^2+1}$, $x = -1$ b) $when\ x = -1, y = -1\ b = -\frac{1}{2}\ y' = m = \frac{3(x^2+1)-2x(3x+1)}{(x^2+1)^2}\ \ \therefore y = \frac{1}{2}x - \frac{1}{2}$

Q Find the first three derivatives of each of the following functions.

a) $y = x^4 - 3x^3 + 16x$

Answer: a) $y' = 4x^3 - 9x^2 + 16,\ y'' = 12x^2 - 18x,\ y''' = 24x - 18$

b) $y = \frac{2x-1}{x}$ b) $y' = x^{-2},\ y'' = -2x^{-3},\ y''' = 6x^{-4}$

Q Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and have a slope 12. *Answer:* $y' = 3x^2\ \ x = \pm 2$

$when\ x = -2, y = -7\ \ and\ b = 17\ \ \therefore y = 12x + 17$

$when\ x = 2, y = 9\ \ and\ \ b = -15\ \ \therefore\ y = 12x - 15$

Q The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in metres.

a) Find the velocity at time t. *Answers:* a) $v = s^1 = f'(t) = 3t^2 - 12t + 9$

b) What is the velocity after 2 s? after 4 s? b) $v(2) = -3\ m/s,\ v(4) = 9\ m/s$

c) When is the particle at rest? c) $v = 3t^2 - 12t + 9\ when\ v = 0, t = 3,1$

d) When is the particle moving in the positive direction? in the negative direction?

d) $pos\ t \in [0,1)\ and\ (3,\infty)\ \ neg\ t \in (1,3)$

e) Find the acceleration at time t and after 4 s. e) $a(t) = 6t - 12,\ a(4) = 12\ m/s^2$

f) When is the particle speeding up? slowing down?

f) $speeding\ up, t \in (2,\infty)\ \ slowing\ down\ t \in (0,2)$

SCO: **C4:** Find derivatives of trigonometric functions. [CN, ME, R, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C4. Find derivatives of trigonometric functions.

Scope and Sequence of Outcomes:

Pre-Calc 12A and 12B	Calculus 12
T6 (PC12A) L1, L2, L3 (PC12B)	**C4. Fi**nd derivatives of trigonometric functions.

ELABORATION

The derivatives of the trigonometric functions $y = \sin x$ and $y = \cos x$ can be derived by using the definition of the derivative using the two fundamental limits,

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \qquad and \qquad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0,$$

the angle sum formulas and the trigonometric identities.

The derivatives of the four other trigonometric functions, tangent, **cosecant, secant** and **cotangent**, can be derived by rewriting each function in terms of $\sin x$ and/or $\cos x$, and then using the Quotient Rule. Students should be reminded that variables in trigonometric functions are always measured in radians. The derivatives of the six trigonometric functions are:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Since students have not studied the chain rule yet, all trigonometric functions cannot be composite functions. L'Hôpital's Rule will be used to evaluate limits, only when other basic methods do not work.

ACHIEVEMENT INDICATORS

- Establish each of the following trigonometric limits, using informal methods:

  $$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

- Derive the derivatives of the six basic trigonometric functions.
- Determine the derivative of a trigonometric function.
- Use the Sandwich Theorem (Squeeze Theorem) to find certain limits indirectly.
- Solve a problem involving the derivative of a trigonometric function.
- Evaluate limits involving trigonometric functions using L'Hôpital's Rule:

  $$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

SCO: **C4:** Find derivatives of trigonometric functions. [CN, ME, R, V]

Suggested Instructional Strategies

- Have students investigate the behaviour of the graph of the derivative of $y = \sin x$ leading them to recognize the graph of $y = \cos x$.
- It may be helpful to review the reciprocal, quotient, Pythagorean, and angle sum identities.
- It is helpful to have students incorporate graphing calculators or other computer-aided software into class explorations.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Find the derivative of each of the following functions.

a) $f(x) = 3x^2 - 2\cos x$

b) $f(x) = \frac{x}{2-\tan x}$

c) $f(x) = x^2 \cos x$

d) $f(x) = \frac{x\sin x}{1+x}$

e) $f(x) = x - 4\csc x + 2\cot x$

f) $f(x) = \csc x \cot x$

Answers: a) $f'(x) = 6x + 2\sin x$

b) $f'(x) = \frac{2-\tan x + x\sec^2 x}{(2-\tan x)^2}$

c) $f'(x) = -x^2 \sin x + 2x\cos x$

d) $f'(x) = \frac{x(1+x)\cos x + \sin x}{(1+x)^2}$

e) $f'(x) = 1 + 4\csc x \cot x - 2\csc^2 x$

f) $f'(x) = -\csc^3 x - \csc x \cot^2 x$

Q Find an equation of the tangent line to the curve $y = 2x\sin x$ at the point where $x = \frac{\pi}{2}$.

Answer: $y' = 2x\cos x + 2\sin x$

$$y' = \left(\frac{\pi}{2}\right) = 2\cdot\frac{\pi}{2}\cdot\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}$$

tangent at $\left(\frac{\pi}{2}, \pi\right)$ $m = 2$

$$y - y_1 = m(x - x_1)$$

$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

$$y = 2x$$

Q Find all the points where the function $f(x) = \sec x$ has a horizontal tangent line.

Answer: $f'(x) = \sec x \tan x = 0$

either $\sec x = 0$ *or* $\tan x = 0$

$x = \emptyset$ $x = \cdots, -2\pi,\ 0,\ 2\pi, \ldots$

$x = 2n\pi,$ where $n \in I$

SCO: **C5:** Apply the Chain Rule to determine the derivative of a function. [C, CN, PS, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

C5. Apply the Chain Rule to determine the derivative of a function.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
	C5. Apply the Chain Rule to determine the derivative of a function.

ELABORATION

The derivative of the function $y = (x+1)^{10}$ can be found by expanding the expression $(x+1)^{10}$ then applying the Sum Rule to the expanded expression. But this is an inefficient method, especially if the outer exponent is very large. Finding the derivative of the function $y = \sin(x+1)$ can be found by using the trigonometry identity for the sine of a sum and then applying the appropriate derivative rules to find the answer, but again the method is inefficient.

The **Chain Rule** can be used to more easily find the derivative of composite functions such as these and many others. It states that if two functions f and g are both differentiable, and $F(x) = (f \circ g)(x) = f[g(x)]$
then $F'(x) = (f \circ g)'(x) = f'[g(x)] \cdot g'(x)$

Because of its structure, the Chain Rule is sometimes referred to as the "Outside-Inside" Rule.

An important case of the Chain Rule applies to a function that is made up of an expression raised to a power. The **Power Chain Rule**, as it is called, states that if n is a real number and $f(u) = u^n$, then $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

Any number of "links" of a chain can be created using the chain rule, where one variable is written in terms of another, the following chain having three links,
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$. It is important to highlight that these are not quotients, but derivatives, though it may appear that common terms "cancel".

For example: To find the derivative of $y = \sin^4(x^2+2x)$

- Let $u = \sin(x^2+2x)$ and substitute $y = u^4$. Find $\frac{dy}{du}$.
- Let $w = (x^2+2x)$, and substitute $u = \sin w$. Find $\frac{du}{dw}$.
- Since $w = x^2+2x$, find $\frac{dw}{dx}$.

To find the derivative of the original function, put the links of the chain together,
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$

Most texts begin by taking a traditional approach for teaching the correct usage of the Chain Rule. First, students are taught to differentiate $y = f[g(x)]$ by setting $u = g(x)$, calculating the two derivatives $f'(u)$ and $g'(x)$, and then applying the **Chain Rule** to obtain $y' = f'(u)g'(x) = f'[g(x)]g'(x)$. This process is then shortened by dispensing with the u and simply referring to $g(x)$ as the **inside function**. This abbreviated process is called the **Outside-Inside Rule**. In applying the Outside-Inside Rule a common mistake when differentiating $f[g(x)]$ is to omit $g'(x)$ in the answer.

It is not always possible for a relation to be written as an explicit function of one variable. In order to find $\frac{dy}{dx}$ in these cases, implicit differentiation must be used. In this method the derivative of each term in the relation is taken with respect to the independent variable x. This means that whenever a y variable is differentiated, it will always be followed by a $\frac{dy}{dx}$ as a result of chain rule. The resulting equation for $\frac{dy}{dx}$ is then solved to get the desired derivative.

For implicit differentiation highlight:
- The final answer could contain both variables and it may not be possible to solve for one particular variable.
- Implicit differentiation is often used to derive rules for differentiating inverse functions.
- In taking derivatives implicitly which require the Product Rule and/or the Chain Rule, a common mistake is for students to forget to use the Product Rule when differentiating the term xy.
- When finding second derivatives using implicit differentiation it is important to emphasize that $\frac{dy}{dx}$ must be replaced by the previously solved derivative formula to find the answer.

When finding tangent and normal line equations students will refer back to *Outcome C3*.

ACHIEVEMENT INDICATORS

- Demonstrate an understanding of the Chain Rule.
- Determine the derivative of a composite function, using the Chain Rule.
- Solve a problem involving the derivative of a composite function.
- Determine the derivative of a relation, using implicit differentiation.
- Determine the equation of the tangent and normal lines to the graph of a relation at a given point.
- Determine the second derivative of a relation, using implicit differentiation.
- Solve problems involving implicit differentiation drawn from a variety of applications.

Suggested Instructional Strategies

- Students should get plenty of practice with the **Chain Rule** so that its use becomes automatic. Students should complete basic problems solved both by expanding and by using the Outside-Inside Rule to have them see that it comes to the same answer. They would then decide which method is more efficient for solving each specific question.

SCO: **C5:** Apply the Chain Rule to determine the derivative of a function. [C, CN, PS, R, V]

Questions (Q) and Activities (Act) for Instruction and Assessment

Q Find the derivative of each of the following functions.

a) $f(x) = (x^4 + 3x^2 - 2)^5$

b) $f(x) = (2x-3)^4(x^2+x+1)^5$

c) $f(x) = \left(\frac{x^2+1}{x^2-1}\right)^3$

d) $f(x) = \frac{x+1}{\sin^2 x}$

e) $Given\ that\ \frac{dy}{du} = 5,\ \frac{dv}{dx} = 2x,\ \frac{du}{dv} = 3,\ Find\ \frac{dy}{dx}$

Answers: a) $f'(x) = 5(x^4 + 3x^2 - 2)^4(4x^3 + 6x)$

b) $f'(x) = (2x-3)^3(x^2+x+1)^4(28x^2 - 12x - 7)$

c) $f'(x) = \frac{-12x(x^2+1)^2}{(x^2-1)^4}$

d) $f'(x) = \frac{\sin x^2 - (x+1)(2\sin x)\cos x}{\sin^4 x}$

$or\ \ f'(x) = \frac{\sin^2 x - (x+1)(\sin 2x)}{\sin^4 x}$

$or\ \ f'(x) = \frac{\sin x - 2(x+1)\cos x}{\sin^3 x}$

e) $\frac{dy}{dx} = 30x$

Q Find the value of $(f\circ g)'(x)$ for each of the following functions at the indicated point.

a) $f(x) = u^{10},\ u = g(x) = 1 + 2x,\ x = 0$

b) $f(x) = \sin u,\ u = g(x) = \cos x,\ x = \frac{\pi}{2}$

Answers: a) $f'(x) = 20(1+2x)^9,\quad f(0) = 20$

b) $f'(x) = \cos(\cos x)(-\sin x),\quad f'\left(\frac{\pi}{2}\right) = -1$

Q For each of the following relations, find $\frac{dy}{dx}$ by implicit differentiation.

a) $9x^2 - y^2 = 1$

b) $x^2 + xy - y^2 = 4$

c) $y\cos x = x^2 + y^2$

d) $y^5 + x^2y^3 = 1 + x^4y$

Answers: a) $\frac{dy}{dx} = \frac{9x}{y}$

b) $\frac{dy}{dx} = \frac{(-2x-y)}{(x-2y)}$

c) $\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}$

d) $\frac{dy}{dx} = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}$

Q For each of the following relations, find the slope of the tangent to the curve at the given point.

a) $x^2 + xy + y^2 = 3,\ P(1,1)$

b) $x^{1/3} + y^{1/3} = 3,\ P(8,1)$

Answers: a) $slope = -1$

b) $slope = -\frac{1}{4}$

Q Determine the point(s) on the graph of $x^2 + 2xy - y^2 = 9$ where the slope of the tangent to the curve is not defined. *Answer:* $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)\ \left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

Q Find the equation of the tangent and the normal lines to the curve $xy + y^3 = 3$ at the point $(2, 1)$. *Answer:* *tangent* $y = -\frac{1}{5}x + \frac{7}{5}$ *normal* $y = 5x - 9$

Q Use implicit differentiation to find $\frac{d^2y}{dx^2}$ for the relation $9x^2 - 4y^2 = 36$.

Answer: $\frac{d^2y}{dx^2} = \frac{36y^2 - 81x^2}{16y^3}$

SCO: **C5:** Apply the Chain Rule to determine the derivative of a function. [C, CN, PS, R, V]

Q Find the derivative of each of the following functions.

a) $f(x) = \sqrt{3x^2 - 1}$ $\quad$ *Answers:* $\quad$ a) $f'(x) = \frac{3x}{\sqrt{3x^2-1}}$

b) $f(x) = (\sin x)^{-2/3}$ $\quad$ b) $f'(x) = -\frac{2}{3}(\sin x)^{-\frac{5}{3}}(\cos x)$

Q Find the equation of the normal line to the curve $s(t) = t\sqrt[3]{2t+5}$ at $t = 11$.

Answer: $\quad y = -\frac{27x}{103} + \frac{3696}{103}$

Q Show that $y'' = -\frac{20x}{y^5}$ where $x^3 + y^3 = 10$.

Answer: $\quad \frac{dy}{dx} = -\frac{x^2}{y^2} \qquad y" = \frac{y^2(-2x) - (-x^2)(2y\frac{dy}{dx})}{y^4}$

Q Suppose y changes twice as fast as u and u changes three times as fast as x. How does the rate of change of y compare with the rate of change of x?

Answer: $\quad \frac{dy}{dt} = 2\frac{du}{dt} \qquad \frac{du}{dt} = 3\frac{dx}{dt} \qquad \therefore \frac{dy}{dt} = 6\frac{dx}{dt}$ *which is 6 times as fast*

SCO **C6: Solve problems involving inverse trigonometric functions.** [CN, ME, R, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C6. Solve problems involving inverse trigonometric functions.

Scope and Sequence of Outcomes:

Pre-Calculus A 120	Calculus 120
RF6-8, T1-T4	**C6.** Solve problems involving inverse trigonometric functions.

ELABORATION

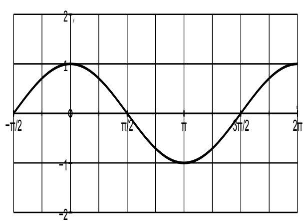
In *Pre-Calculus A 120* students were introduced to inverse functions and logarithmic functions as the inverse of exponential functions. They also explored the six trigonometric ratios with reference to the unit circle, expressed in degrees and radians. They graphed and analyzed sine, cosine and tangent functions, and defined the domains and ranges of these functions.

For this outcome students will review the graphs, domains and ranges of the three primary trigonometric functions and then will explore their **inverse functions** and the restrictions placed on the domains and ranges of these functions.

The notation for the **inverse trigonometric functions** of the primary trig functions is $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$, also referred to as $\arcsin x$, $\arccos x$ and $\arctan x$. Students can be reminded of the inverse trig function keys on their calculators and their use of them to solve for the value of an angle. They may also recall that their calculators only give them one solution and this can now be related to the restricted ranges in inverse functions.

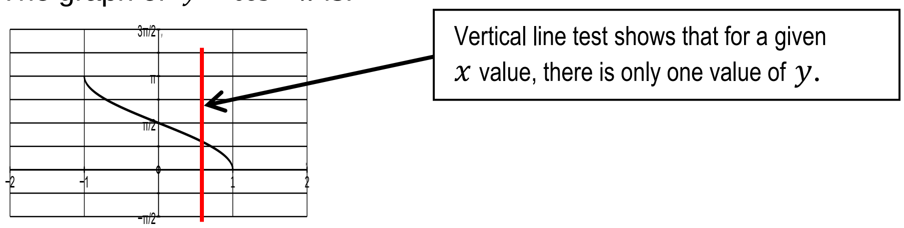
For example, the inverse function of $y = \cos x$ can be written as $x = \cos y$, or $y = \cos^{-1} x$ where all values of x and y are switched for the new function. However, for the inverse function to retain the characteristics of a function, for every value of x there must be only one value for y (vertical line test). To produce a new function that does have an inverse function the domain of $y = \cos x$ is restricted as shown below.

The graph of $y = \cos x$ is:



$Domain = \{x | x \in R\}$ $\quad$ $Range = \{y | -1 \le y \le 1, y \in R\}$

The graph of $y = \cos^{-1} x$ is:



$Domain = \{x | -1 \le x \le 1, x \in R\}$ $\quad$ $Range = \{y | 0 \le y \le \pi, y \in R\}$

SCO **C6: Solve problems involving inverse trigonometric functions.** [CN, ME, R, V]

The inverse trigonometric functions are restricted as follows:

FUNCTION	*DOMAIN*	*RANGE*
$y = \sin^{-1} x = \arcsin x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x = \arccos x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x = \arctan x$	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

To find the derivatives of the primary inverse trigonometric functions, the function can be expressed as its inverse and then implicit differentiation can be used. For example $y = \sin^{-1} x$, can be written as $x = \sin y$ and then implicit differentiation can be used to solve for the derivative:

$if \ \ \sin y = x$

$\frac{d}{dx}(\sin y) = \frac{d}{dx} x$

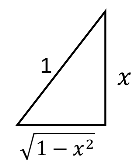
$\cos y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-(\sin y)^2}} = \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

OR

$if \ \ \sin y = x$



$then \ \ \cos y = \sqrt{1-x^2}$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

The derivative of $\arccos x$ is $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

The derivative of $\arctan x$ is $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

Inverse trig functions should not be confused with reciprocal trig functions that students will be familiar with from *Pre-Calculus A 120*. The exponential notation is similar but distinct: $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$ whereas $\sin^{-1} x = arc \sin x$.

ACHIEVEMENT INDICATORS

- Explain the relationship between the primary trigonometric functions and the corresponding inverse trigonometric function.
- Explain why inverse trigonometric functions have restricted domains and ranges.
- Determine the exact value of an expression involving an inverse trigonometric function.
- Simplify an expression involving an inverse trigonometric function.
- Determine the domain of an inverse trigonometric function.
- Sketch the graph of an inverse trigonometric function.
- Determine the derivative of inverse trigonometric functions including composites.
- Solve problems involving the derivative of an inverse trigonometric function.

SCO **C6: Solve problems involving inverse trigonometric functions.** [CN, ME, R, V]

Suggested Instructional Strategies

- Students should be given enough time to ensure they are comfortable working with inverse trigonometric functions.
- Students should understand how the derivatives of inverse functions are derived. An understanding of these concepts is much more useful than memorizing formulas.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Act Have students graph the function $y = \sin x$. How can the domain of this function be restricted so that it passes the horizontal line test? Can this be done in more than one way? Why is the 4th and 1st quadrant commonly chosen to represent the inverse? *(This type of question can be repeated for all trig function with the quadrants adjusted accordingly.)*

Q Solve each of the following equations in the specified interval. Round off all answers to three decimal places.

a) $\cos x = 0.4$, $0 \le x < \pi$
b) $\sin x = -\frac{3}{4}$, $-\frac{\pi}{2} \le x < \frac{\pi}{2}$
c) $\tan x = 2$, $-\frac{\pi}{2} \le x < \frac{\pi}{2}$

Answers: a) $x = 1.159$ b) $x = -0.848$ c) $x = 1.107$

Q Find the exact value of each expression.

a) $\cos^{-1}(0.5)$
b) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
c) $\tan^{-1}(-\sqrt{3})$
d) $\cos\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$
e) $\tan\left[\sin^{-1}\left(\frac{2}{5}\right)\right]$

Answers: a) $\frac{\pi}{3}$ b) $\frac{\pi}{4}$ c) $-\frac{\pi}{3}$ d) $\frac{1}{2}$ e) $\frac{2\sqrt{21}}{21}$

Q Simplify each of the following expressions.

a) $\cos(\sin^{-1} x)$
b) $\tan(\cos^{-1} x)$

Answers: a) $\sqrt{1 - x^2}$ b) $\frac{\sqrt{1-x^2}}{x}$

Q Find the derivative of each of the following functions.

a) $y = \tan^{-1}(x^2)$
b) $y = \tan^{-1}(\cos x)$
c) $y = x\sin^{-1} x + \sqrt{1 - x^2}$
d) $y = \sin^{-1}(2x + 1)$

Answers: a) $y' = \frac{2x}{1+x^4}$ b) $y' = \frac{-\sin x}{1+\cos^2 x}$ c) $y' = \sin^{-1} x$ d) $y' = \frac{1}{\sqrt{-(x^2+x)}}$

Q Find the equation of the tangent line to the curve $y = \cos^{-1} x$ at the point where $x = \frac{1}{2}$.

Answer: $y = \frac{-2\sqrt{3}}{3}x + \frac{\sqrt{3}+\pi}{3}$

Q Differentiate $y^2 \sin x = \tan^{-1}(x) - y$ with respect to x.

Answer: $\frac{dy}{dx} = \left(\frac{1}{2y\sin x+1}\right)\left[\frac{1}{1+x^2} - y^2\cos x\right]$

Q Why do the vertical asymptotes of the tangent function become the horizontal asymptotes of the inverse tangent function? Explain your reasoning.

Answer: Because the x features become y features in the inverse function.

SCO: **C7. Find limits and derivatives of exponential and logarithmic functions.** [CN, ME, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

C7. Find limits and derivatives of exponential and logarithmic functions.

Scope and Sequence of Outcomes:

Pre-Calculus A 120	Calculus 120
RF 7-11	**C7.** Find limits and derivatives of exponential and logarithmic functions.

ELABORATION

In *Pre-Calculus A 120* students explored exponential and logarithmic functions and their graphs. A brief review of the properties of logarithms is appropriate, since students will need a firm understanding of these concepts to move on.

For this outcome, the exponential limit will be established using the definition of the derivative:

$$\lim_{h\to 0}\frac{e^{h}-1}{h} = 1$$

The derivatives of the two basic exponential functions are $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}a^x = a^x \ln a$.

Exploring the rate of change for exponential functions involves solving problems related to exponential growth and decay.

The derivatives of the two basic logarithmic functions are $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$. These rules can be expanded to handle any chain rule applications involving composite functions.

Logarithmic differentiation involves taking the natural logarithm of both sides, taking the derivative of each side with respect to x, and then solving for $\frac{dy}{dx}$. This creates an expression which has two different variables in the derivative. To eliminate one variable, the original function is substituted for y to get the final answer in terms of x only.

Logarithmic differentiation can be used to find the derivative of a function containing a variable in the exponent. It can also be used for complicated composite functions.

Limits will be evaluated using *L'Hôpital's Rule* only when other basic methods do not work.

SCO: **C7. Find limits and derivatives of exponential and logarithmic functions.** [CN, ME, R, V]

ACHIEVEMENT INDICATORS

- Establish the exponential limit using informal methods.
  $$\lim_{h\to 0}\frac{e^h-1}{h}=1$$
- Determine the derivatives of the exponential functions $y=a^x$ and $y=e^x$, and of the logarithmic functions $y=\log_a x$ and $y=\ln x$.
- Determine the derivative of composite exponential and logarithmic functions.
- Determine the derivative of a function using logarithmic differentiation.
- Solve a problem involving the derivative of an exponential or a logarithmic function. (e.g. exponential growth and decay problems).
- Evaluate limits involving logarithmic functions using L'Hôpital's Rule.
  $$\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}$$

Suggested Instructional Strategies

- Have a discussion with students regarding the graph of $y=e^x$ and what properties the derivative of this function ought to have.
- Have students work first with a^x and e^x before they go on to equations with $\log x$ and $\ln x$.
- Ensure that students understand why the derivative of $y=x^x$ cannot be found directly using the Power Rule.
- Students should be encouraged to use the properties of logarithms to determine the derivatives of equations such as $y=\ln\left(\frac{x}{2x+5}\right)^3$ which can be rewritten as $y=3\ln x-3\ln(2x+5)$.
- Additional logarithmic differentiation examples and questions to supplement the prescribed text can be found on various online sites such as: Log Differentiation 1 Log Differentiation 2 Log Differentiation 3

SCO: **C7. Find limits and derivatives of exponential and logarithmic functions.** [CN, ME, R, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Find the derivative of each of the following functions.

a) $f(x) = 5e^{2x}$

b) $f(x) = x^2e^{x-2}$

c) $f(x) = (x^3 + 2x)e^x$

d) $f(x) = e^x \sin x$

e) $f(x) = \ln(2x + 3)$

f) $f(x) = x \ln x - 3x^2$

g) $f(x) = 7^{3x^2}$

h) $f(x) = \log_5(3x - 1)$

i) $y = \ln(\sin 2x)$

j) $y = \ln\left(\frac{x}{2x+5}\right)^3$

Answers:
a) $f'(x) = 10e^{2x}$
b) $f'(x) = xe^{x-2}(x + 2)$
c) $f'(x) = e^x(x^3 + 3x^2 + 2x + 2)$
d) $f'(x) = e^x(\cos x + \sin x)$
e) $f'(x) = \frac{2}{2x+3}$
f) $f'(x) = 1 - 6x + \ln x$
g) $f'(x) = 6x(\ln 7) \cdot 7^{3x^2}$
h) $f'(x) = \frac{3}{(3x-1)\ln 5}$
i) $y' = 2\cot 2x$
j) $y = \frac{3}{x} - \frac{6}{2x+5}$ *OR* $y' = \frac{15}{x(2x+5)}$

Q Find the equation of the tangent line to the curve $y = e^{5x}$ at the point where $x = 0$.

Answer: $y = 5x + 1$

Q Have students compute the derivative $y = e^x \cdot e^{x^2}$ in two different ways.

Answer: *Product Rule:* $y' = e^x \cdot 2xe^{x^2} + e^{x^2} \cdot e^x = 2xe^{x^2+x} + e^{x^2+x} = e^{x^2+x}(2x + 1)$
Simplify First: $y = e^x \cdot e^{x^2} = e^{x^2+x}$ $\quad y' = (2x + 1)e^{x^2+x}$

Q Use logarithmic differentiation to find the derivative of each of the following functions

a) $f(x) = x^{\sin x}$

b) $f(x) = \frac{(3x+1)^4(x-1)^6}{\sqrt{6x-5}}$

Answers:
a) $\frac{dy}{dx} = x^{\sin x}\left[\frac{1}{x}\sin x + (\cos x)(\ln x)\right]$
b) $\frac{dy}{dx} = \frac{(3x+1)^4(x-1)^6}{\sqrt{6x-5}}\left[\frac{12}{3x+1} + \frac{6}{x-1} - \frac{3}{6x-5}\right]$

Q a) Determine the equation of the tangent to the curve $y = x\ln x$ at $x = 1$.

b) How would the answer change if the equation was $y = x\log x$ at $x = 1$?

Answers: a) $y = x - 1$ b) $y = \frac{1}{\ln 10}x - \frac{1}{\ln 10}$

SCO: **C8: Use calculus techniques to sketch the graph of a function.** [C, CN, PS, R, T, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C8. Use calculus techniques to sketch the graph of a function.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
	C8. Use calculus techniques to sketch the graph of a function.

ELABORATION

For this outcome a firm grasp of terminology will be important. To sketch the graph of a function $y = f(x)$, the domain, intercepts, symmetry, asymptotes, critical values, intervals for increase and decrease, local maximum and minimum, and intervals of concavity can be determined as described below.

1) Domain and intercepts:
   - The Domain is the set of values of x for which $y = f(x)$ is defined.
   - The y-intercept is found at $f(0)$.
   - The x-intercepts are found by solving for x when $y = 0$, using factoring (including the quadratic equation), or using technology to find approximate values.

2) Symmetry:
   - If $f(-x) = f(x)$ for all x in the domain, then f is an even function, and the curve is symmetric about the y-axis.
   - If $f(-x) = -f(x)$ for all x in the domain, then f is an odd function, and the curve is symmetric about the origin.
   - If $f(x + p) = f(x)$ for all x in the domain, where p is a positive constant, then f is periodic, which means that the curve repeats over the interval p.

3) Asymptotes:
   - If there exists a line $y = b$ such that
     $\lim_{x\to\infty} f(x) = b$ and/or $\lim_{x\to-\infty} f(x) = b$
     then $y = b$ is the horizontal asymptote of the curve $y = f(x)$.
   - If there exists a line $x = a$ such that
     $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$
     then $x = a$ is the vertical asymptote of the curve $y = f(x)$.

4) Intervals of increase or decrease:
   - The critical values of $y = f(x)$ can be found using its derivative $y = f'(x)$, the values where $f'(x) = 0$, or where $f'(x)$ does not exist. The function will be increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. It is important that students consider points where the derivative is undefined.

     Note *that some resources describe intervals in which a function increases and decreases, as closed intervals and others as open intervals.*

SCO: **C8: Use calculus techniques to sketch the graph of a function.** [C, CN, PS, R, T, V]

- The *Mean Value Theorem* connects the average rate of change of a function over an interval with the instantaneous rate of change of the function at a point within the interval. It states that if $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which: $f'(c) = \frac{f(b)-f(a)}{b-a}$.

5) Local maximum and local minimum values:

- The *Extreme Value Theorem* states that If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval.
- The *First Derivative Test* is used to determine whether the critical values are local maximum or local minimum values.
  - If f' changes from negative to positive at c and $f(c)$ exists, then $f(c)$ is a local minimum.
  - If f' changes from positive to negative at c and $f(c)$ exists, then $f(c)$ is a local maximum.
  - If f' does not change sign at a critical point c, then f has no local extreme value at c.
- The *Second Derivative Test* can also be used to determine if a point is a local minimum or maximum. However, sometimes this proves too complicated or lengthy to find algebraically, and the *First Derivative Test* is easier and more appropriate.

6) Intervals of concavity and the points of inflection:

- The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Some students will identify any points for which $f''(x) = 0$ as points of inflection. It is important to note that a change in concavity must exist in order for a function to have a point of inflection.
- An inflection point will occur where $f"(x) = 0$, providing $f(x)$ exists, and the concavity is different on either side of the point.
- Number lines and charts are sometimes used to describe increasing and decreasing situations as well as concavity. Students will need to be able to explain what these charts and number lines are telling them.

Technology can be used as a check in these sections. Since confirmation of what is seen comes from analysis, it is important to incorporate previously acquired algebraic skills into the study of calculus.

Students should understand the connection between the graph of a function, and the graphs of its first and second derivatives. The analysis of the first and second derivatives establishes all the important features suggested by the graph on a graphing calculator.

SCO: **C8: Use calculus techniques to sketch the graph of a function.** [C, CN, PS, R, T, V]

ACHIEVEMENT INDICATORS

- Determine the local and global (absolute) extreme values of a function.
- Demonstrate an understanding of the *Extreme Value Theorem*.
- Determine the critical and stationary points of a function.
- Demonstrate an understanding of the *Mean Value Theorem*.
- Determine the intervals on which a function is increasing and decreasing.
- Use the *First Derivative Test* to locate the local extrema of a function.
- Use the *First* or *Second Derivative Test* to classify the local extrema of a function.
- Use the *Concavity Test* (second derivative) to determine the intervals of concavity of a function.
- Determine the points of inflection of a function.
- Determine asymptotic and end behaviour of a function using limits.
- Determine the key features of the graph of a function, using the techniques of differential calculus, and use these features to sketch the graph without technology.
- Sketch the graph of a function, using information about its derivative.

Suggested Instructional Strategies

- Sketch an arbitrary function and discuss the local and absolute maxima and minima of the function. Students can understand the definitions of these concepts more easily if they have an intuitive introduction to their meaning.
- Present several graphs of functions to the students, have them discuss their derivatives and explain how they relate to the extreme values of each function.
- When giving questions to students on curve sketching, provide both the function and its derivatives for some questions to allow them to easily mark what they find.

SCO: **C8: Use calculus techniques to sketch the graph of a function.** [C, CN, PS, R, T, V]

Suggested Questions (Q) and Activities (Act)

Act Graphs could be posted around the room and cards given to students with various pieces of information given to describe the function, its increase and decrease characteristics, intercepts, asymptotes, symmetry, concavity, domain, range, extrema, end behavior, etc. Students must find which graphs would pertain to their info card.

Q Sketch the graph of a continuous function that satisfies all of the following conditions:

$f(-3) = f(0) = f(3) = 0,\ f(-1) = f(1) = -2,\ f'(-1) = f'(1) = 0, f'(0) = 0$

$f'(x) < 0$ for $x < -1$ and $0 < x < 1$

$f'(x) > 0$ for $-1 < x < 0$ and $x > 1$

$f''(x) > 0$ for $x < -\frac{1}{2},\ \frac{1}{2} < x < -\frac{1}{2}$

$f''(x) < 0$ for $-\frac{1}{2} < x < \frac{1}{2},\ x > 3$

$\lim_{x\to\infty} f(x) = 1,\quad \lim_{x\to-\infty} f(x) = \infty$

Answer: $x\ int = 0, 3,\ and\ -3,\quad y\ int = 0,\quad pts. (-1,-2)\ and\ (1,-2),\quad \min at\ x = -1,$
$\lim_{x\to\infty} f(x) = 1,\quad \lim_{x\to-\infty} = \infty$

Q For each of the following functions, use analytic methods to find the absolute and local extreme values of the function on the given interval.

a) $f(x) = 1 + (x+1)^2,\ -2 \le x \le 5$ *Answers*: a) $(-1,1)$ *absolute min*, $(5,37)$ *absolute max*

b) $f(x) = \frac{1}{x},\ x \ge 1$ b) $(1,1)$ *absolute max*

c) $f(x) = 3x - x^3,\ -3 \le x \le 3$ c) $(-3,18)$ *absolute max*, $(3,18)$ *absolute min*, $(-1,-2)$ *local min*, $(1,2)$ *local max*

d) $f(x) = \cos x,\ -\pi \le x \le \pi$ d) min *at* $(-\pi,-1)$ *and* $(\pi,-1)$, max *at* $(0,1)$

Q Find all absolute and local extreme values of each of the following functions.

a) $f(x) = 4 + 6x - 3x^2$ *Answers*: a) $(1,7)$ *local max*

b) $f(x) = \frac{x-1}{x^2-x+1}$ b) $(0,-1)$ *local min*, $\left(2,\frac{1}{3}\right)$ *local max*

c) $y = x^3 + 3x^2 - 1$ c) $(0,-1)$ *local min*, $(-2,3)$ *local max, no absolute max or min*

Q For each of the following functions, use analytic methods to find
- the local extrema,
- the intervals on which the function is increasing,
- the intervals on which the function is decreasing.

a) $f(x) = 2x^3 + 3x^2 - 36x$
Answer: *local* $\min(2,-44)$ $\max(-3,81)$, *increasing* $x \in (-\infty,-3) \cup (2,\infty)$, *decreasing* $x \in (-3,2)$

b) $f(x) = x^4 - 2x^2 + 3$
Answer: *local* $\min(1,2),(-1,2)$ $\max(0,3)$, *increasing* $x \in (-1,0) \cup (1,\infty)$, *decrasing* $x \in (-\infty,-1) \cup (0,1)$

c) $f(x) = \frac{x}{x^2+1}$
Answer: *local* $\min\left(-1,-\frac{1}{2}\right)$ $\max\left(1,\frac{1}{2}\right)$, *increasing* $x \in (-1,1)$, *decreasing* $x \in (-\infty,-1) \cup (1,\infty)$

d) $f(x) = x\sqrt{x-9}$
Answer: *absolute* $\min(9,0)$, *increasing* $x \in (9,\infty)$

SCO: **C8: Use calculus techniques to sketch the graph of a function.** [C, CN, PS, R, T, V]

Q Sketch a graph of a differentiable function $y = f(x)$ that has a local maximum at $(0, 1)$ and absolute minima at $(-1, 0)$ and $(1, 0)$.

Q For each of the following functions, use analytic methods to find
- the points of inflection,
- the intervals on which the function is concave up,
- the intervals on which the function is concave down.

a) $f(x) = 8x - x^3$ $\quad$ *Answers:* $\quad a)\ pt.infl.\,(0,0), concave\ up\ x < 0, concave\ down\ x > 0$

b) $f(x) = 4x^3 - 3x^2 - 6x + 1$ $\quad b)\ pt.infl.\left(\frac{1}{4}, \frac{-5}{8}\right), concave\ up\ x > \frac{1}{4}, concave\ down\ x < \frac{1}{4}$

Q Use analytic methods to sketch the graph of each of the following functions.

a) $f(x) = 2 + 3x^2 - x^3$

$Answer: \min(0,2), \max(2,6), pt.infl.\,(1,4), decrease\ x < 0, x > 2, increase\ 0 < x < 2,$
$concave\ up\ x < 1, concave\ down\ x > 1$

b) $f(x) = \frac{x}{x^2-9}$

$Answer: no\ \max\ or\ \min\ , pt.infl.\,(0,0), always\ decreasing, concave\ up\ -3 < x < 0\ and\ x > 3,$
$concave\ down\ x < -3, 0 < x < 3, vertical\ asymptotes\ at\ x = 3, -3, horizontal\ asymptote\ y = 0$

c) $f(x) = (x - 3)\sqrt{x}$

$Answer: Domain\ x > 0, \min(1,-2), pt.infl.\,(0,0), decrease\ 0 < x < 1, con.up\ x > 0, con.down\ never$

Q Consider a differentiable function $f(x)$ with a domain of all positive real numbers, and for which $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

a) Find the x-coordinate of the critical point of f. Determine whether the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer. $Answers:\ a)\ x = 4, relative\ max$

b) Find all intervals on which the graph of f is concave down. Justify your answer.
$b)\ concave\ down\ x > 0$

c) Sketch a possible graph to represent $f(x)$, given $f(1) = 2$.

Q In your journal respond to:

Anne stated: "for concave up curves, the function must be increasing". Is she correct and explain your reasoning.

$Answer:\ She\ is\ incorrect\ as\ the\ first\ half\ of\ the\ curve\ is\ decreasing\ to\ the\ minimum\ point, then\ increasing.$

Q Have students write an explanation for the following situations.

a) $f(x) = x^3$ and $f'(x) = 0$ at $x = 0$, but there are no minimum or maximum values at $x = 0$.

b) $f(x) = x^4$ and $f''(x) = 0$ at $x = 0$, but there is no inflection point at $x = 0$.

$Answer: a)\ inflection\ point\ at\ (0,0), no\ \max\ or\ \min \quad b)\ minimum\ at\ (0,0), no\ inflection\ point$

SCO: **C9: Use calculus techniques to solve optimization problems.** [C, CN, ME, PS, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

C9. Use calculus techniques to solve optimization problems.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
	C9. Use calculus techniques to solve optimization problems.

ELABORATION

A problem which seeks to find the maximum or minimum answer to a situation involving continuous variables is called an **optimization problem**. The steps in solving an optimization problem are as follows:

A. Understand the problem. The first step is to read the problem carefully until it is clearly understood. *What is the unknown? What are the given quantities? What are the given conditions?*

B. Draw a diagram, if necessary. In many optimization problems, it is useful to draw a diagram and identify the given and required quantities on the diagram.

C. Introduce notation. Assign a symbol to the quantity that is to be maximized or minimized, and to other unknown quantities. It may be helpful to chose labels that relate to the value such as, P for profit, or r for radius. Label the diagram with these symbols.

D. Express the objective function in terms of the symbols introduced above. This is the function that ultimately will be maximized or minimized.

E. Express the objective function in terms of only one variable. If necessary, rewrite the objective function so that the right-hand side contains only one variable. In order to do this, look for relationships among the variables that were introduced above. Each of these relationships is called a **parameter.** Use these parameters to eliminate all but one of the variables on the right hand side of the objective function. Also, note the domain of this function in order to eliminate any possible extraneous solutions.

F. Use calculus techniques to find the absolute maximum or minimum value of the objective function. Find the critical values of the objective function by taking its derivative and determining its critical values. Use the First Derivative Test to determine whether each critical value is a local maximum or minimum value. Then, use this information to find either the absolute maximum or absolute minimum value of the objective function. In optimization problems, students may overlook endpoints as possible candidates for optimal values, or use solutions that are outside of the domain of the input value.

G. Interpret the solution. Ensure that the solution is in the domain of the objective function. If it is, then translate your mathematical result into the problem setting (with appropriate units).

SCO: **C9: Use calculus techniques to solve optimization problems.** [C, CN, ME, PS, R]

ACHIEVEMENT INDICATORS

- Determine the equation of the objective function to be optimized in an optimization problem.
- Determine the equations of any parameters necessary in an optimization problem.
- Solve an optimization problem drawn from a variety of applications, using calculus techniques.
- Interpret the solution(s) to an optimization problem.

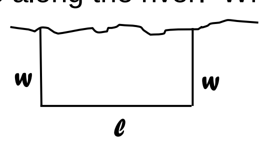
Suggested Instructional Strategies

- Ask students to suggest situations in which one might want to find the minimum or maximum values of a function.
- Students traditionally have difficulty with optimization problems, particularly with the formation of the function to be optimized and the determination of the domain for the problem situation. If this is the case, stress the six-step strategy for solving optimization problems with the students.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares? $Answer\text{: } x = 8$

Q A farmer has $2400\ m$ of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



$Answer\text{: } w = 600,\ l = 1200m$

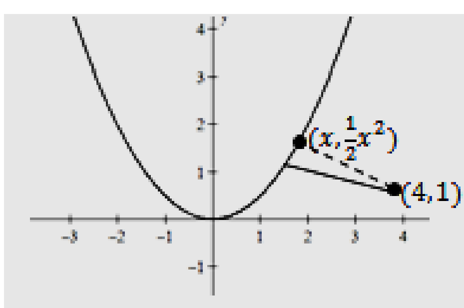
Q A store has been selling 200 Blu-Ray disc players a week at $\$350$ each. A market survey indicates that for each $\$10$ rebate offered to buyers, the number of units sold will increase by 20 per week. At what price should the store sell its Blu-Ray disc players to maximize its revenue? $Answer\text{: } \$225\ each$

Q Find the point on the parabola $y = \frac{1}{2}x^2$ that is closest to the point $(4, 1)$.

$Answer\text{: } d^2 = (4-x)^2 + \left(1 - \frac{1}{2}x^2\right)^2$

$d^{2\prime} = 2(4-x)(-1) + 2\left(1 - \frac{1}{2}x^2\right)(-x)$

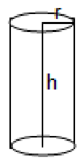
$x = 2 \qquad f(2) = 2 \qquad \therefore\ closest\ point\ on\ parabola\ to\ (4,1)\ \ is\ (2,2)$



SCO: **C9: Use calculus techniques to solve optimization problems.** [C, CN, ME, PS, R]

Q A cylindrical can is to be made to hold 1 L of oil, which has a volume of 1000 cm^3. Find the dimensions that will minimize the cost of the metal to manufacture the can. Round off the answers to one decimal place.

$V = \pi r^2 h \quad S.A. = 2\pi r^2 + 2\pi r h$ $\quad$ *Answer*: $r = \sqrt[3]{\frac{500}{\pi}}$ $\quad h = \frac{1000}{\left(\sqrt[3]{\pi}\right)\left(\sqrt[3]{500^2}\right)}$



Q A rock is thrown upwards from a $200\ m$ cliff. If the height of the rock above the bottom of the cliff is given by $h(t) = -5t^2 + 20t + 200$, determine the maximum height of the rock. $\quad$ *Answer*: max $height = 220\ m$

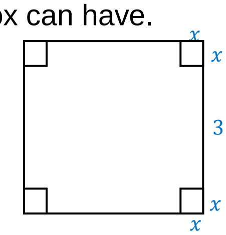
Q A box with an open top is to be constructed from a square piece of cardboard, $3\ ft$ wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

Answer: $V = (3-2x)(3-2x)x$

$V' = 3(4x^2 - 2x - 6x + 3) = 0$

$x = \frac{1}{2}$ or $x = \cancel{\frac{3}{2}}$

$V = 2ft^3$



SCO: **C10. Use linearization (and Newton's Method - optional) to solve problems.** [PS, CN, R, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C10. Use linearization to solve problems.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
	C10. Use linearization to solve problems (Optional - Newton's Method)

ELABORATION

If f is differentiable at $x = a$, the equation of the tangent line, $L(x) = f(a) + f'(a)(x - a)$, is defined as the **linearization** of f at a. Linearization can be used to find an approximation of a value that is difficult to calculate by hand but is close to a known value, such as $\sqrt{145}$ or $\cos 3.1$. Also, linearizations can provide useful estimates when dealing with complicated functions, especially if a calculator is not available.

Leibniz (http://en.wikipedia.org/wiki/Gottfried_Wilhelm_Leibniz) used the notation $\frac{dy}{dx}$ to represent the derivative of y with respect to x. The notation looks like a quotient of real numbers, but it is really a limit of quotients in which both numerator and denominator are infinitesimally close to zero.

Since $\frac{dy}{dx} = f'(x)$ and $\frac{dy}{dx}$ acts as a quotient, we can define the **differential** dy as $dy = f'(x)dx$. This implies that dy is a dependent variable that depends on both x and dx. The differential, dy, represents the difference between the linearization and the true function value at a given point.

Newton's Method is a numerical technique for approximating a zero of a function with the zeros of its linearizations. This can be a fast, efficient way to approximate roots of differentiable functions. Under favourable circumstances, the zeros of the linearizations converge rapidly to an accurate approximation. Many calculators use this method to find zeros because it applies to a wide range of functions and gets accurate results in only a few steps. Although significant time can be spent teaching this method, it is recommended that no more than one class period be spent to cover this briefly:

1) Guess a first approximation to a solution of the equation $f(x) = 0$, as close to the actual solution as possible.
2) Use the first approximation to get a second, the second approximation to get a third, and so on, using the formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

When using Newton's Method to find the zeros of a function, some students may stop after finding one zero or may not choose appropriate values for the initial guess x_1. If $f'(x_n) = 0$, Newton's Method will fail because the formula will involve division by zero, and another starting value will need to be chosen. This method will also fail when the starting value is too far from the actual solution, and a closer starting value will need to be chosen. There are other less common instances in which Newton's Method will not work, depending on the complexity of the function.

SCO: **C10. Use linearization (and Newton's Method - optional) to solve problems.** [PS, CN, R, V]

ACHIEVEMENT INDICATORS

- Use linearization to approximate a numerical expression.
- Solve linearization problems drawn from a variety of applications.
- **Optional** Determine the differential of a function.
- **Optional** Use Newton's Method to approximate the solution(s) of an equation.

Suggested Instructional Strategies

- It is important to note the connection between linearization and estimates of change. The differential estimate of the change in f when x changes by dx is based on a linearization of f.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q For each of the following functions, find the linearization at the given point,

a) $f(x) = \sqrt{x+3},\ \ a = 1$ $\quad$ *Answers:* $a)\ L(x) = 2 + \frac{1}{4}(x-1)$

b) $f(x) = \cos x,\ \ a = \frac{\pi}{4}$ $\quad$ $b)\ L(x) = \frac{\sqrt{2}}{2} + \frac{-\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$

Q Approximate each root by using a linearization centered at an appropriate nearby number. Round off all answers to four decimal places.

a) $\sqrt{120}$ $\quad$ *Answers:* $a)\ L(120) = 11 - \frac{1}{22} \cong 10.9545$

b) $\sqrt[3]{126}$ $\quad$ $b)\ L(126) = 5 + \frac{1}{75} \cong 5.0133$

Q For each of the following
- find dy,
- evaluate dy for the given values of x and dx, to four decimal places.

a) $y = x^2 - 4x,\ x = 3,\ dx = 0.1$ $\quad$ *Answers:* $a)\ dy = 0.2$

b) $y = \frac{x}{x+1},\ x = 1,\ dx = 0.05$ $\quad$ $b)\ dy = 0.0125$

Q Use Newton's Method to determine all real solutions for the equation, $x^3 - 2x - 5 = 0$. Round off all answers to four decimal places.

Answer: 2.0946

SCO: C11. Solve problems involving related rates. [C, CN, ME, PS, R]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C11. Solve problems involving related rates.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
	C11. Solve problems involving related rates.

ELABORATION

In related rates problems the rate of change of one quantity is given and the rate of change of a related quantity is found. This is done by finding an equation that relates the two quantities, and then the chain rule is used to differentiate both sides of the equation with respect to time. The following steps are followed:

A. Understand the problem. In particular, identify the variable whose rate of change is unknown and the variable(s) whose rate(s) of change is(are) known.

B. Develop a mathematical model of the problem. Draw a diagram and label the parts that are important to the problem with symbols that have been assigned to each quantity. Be sure to distinguish constant quantities from variables that change over time. Only constant quantities can be assigned numerical values at the start.

C. Write an equation relating the variable whose rate of change is unknown with the variable(s) whose rate(s) of change is(are) known. The formula is often geometric, but it could originate as a scientific application.

D. Differentiate both sides of the equation implicitly with respect to time, t. Follow all the differentiation rules. The Chain Rule will be especially critical, as it will be necessary when differentiating with respect to time, t.

E. Substitute values for any quantities that depend on time. It is only safe to do this after the differentiation step. Substituting too soon makes changeable variables behave like constants, with zero derivatives.

F. Interpret the solution. Translate the mathematical result into the setting of the problem (with appropriate units), and decide whether the result makes sense.

Students should distinguish between optimization problems and related rates problem. It is helpful to remind students that related rates problems relate to time and optimization will use wording to suggest maximum or minimum quantities.

The most common student error in solving related rate problems is substituting a value for a variable too early, making it impossible to take the appropriate derivatives. Emphasize that evaluation is the final step in solving a related rates problem.

SCO: C11. Solve problems involving related rates. [C, CN, ME, PS, R]

ACHIEVEMENT INDICATORS

- Develop a mathematical model for a related rates problem.
- Solve problems involving related rates, drawn from a variety of applications.
- Interpret the solution to a related rates problem.

Suggested Instructional Strategies

- The Chain Rule and implicit differentiation should be reviewed as a mastery of these topics is crucial for success in solving related rate problems where there is not a direct functional relationship between two quantities.
- The six-step strategy for solving related rate problems should be presented and modelled with the students.
- Web Link: This site contains an applet showing how the volume and radius of a snowball changes with respect to time.
  http://www.mathopenref.com/calcsnowballproblem.html

SCO: C11. Solve problems involving related rates. [C, CN, ME, PS, R]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Using $xy + y^2 = x$, find $\frac{dx}{dt}$ when $y = 2$ and $\frac{dy}{dt} = -3$.

$Answer$: $\frac{dx}{dt} = \frac{x+2y}{1-y} \cdot \frac{dy}{dt}$ $\left.\frac{dx}{dt}\right|_{y=2} = 0$

Q Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ cm^3/s$. How quickly is the radius of the balloon increasing when the diameter is $50\ cm$? $Answer$: $\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dv}{dt}$ $\left.\frac{dr}{dt}\right|_{d=50} = \frac{1}{25\pi}\ cm/s$

Q A ladder $10\ ft$ long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1\ ft/s$, how quickly is the top of the ladder sliding down the wall when the bottom of the ladder is $6\ ft$ from the wall?

$Answer$: $\frac{dy}{dt} = -\frac{x}{y} = \frac{dx}{dt}$ $\left.\frac{dy}{dt}\right|_{x=6} = -\frac{3}{4}\ ft/s$

Q $Car\ A$ is travelling west at $50\ mph$ and $Car\ B$ is travelling north at $60\ mph$. Both are headed towards the common intersection of two roads. At what rate are the cars approaching each other when $Car\ A$ is $0.3\ miles$ and $Car\ B$ is $0.4\ miles$ from the intersection? $Answer$: $\frac{dc}{dt} = \frac{1}{c}\left[a\frac{da}{dt} + b\frac{db}{dt}\right]$ $\left.\frac{dc}{dt}\right|_{\substack{a=0.3 \\ b=0.4}} = -78\ mph$

Q A water tank has the shape of an inverted circular cone with base radius $2\ m$ and height $4\ m$. If water is being pumped into the tank at a rate of $2\ m^3/min$, find the rate at which the water level is rising when the water is $3\ m$ deep.

$Answer$: $\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dv}{dt}$ $\left.\frac{dh}{dt}\right|_{h=3} = \frac{8}{9\pi}\ m/min$

Q A man walks along a straight path at a speed of $4\ ft/s$. A searchlight is located on the ground $20\ ft$ from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is $15\ ft$ from the point on the path closest to the searchlight? $Answer$: $\frac{d\theta}{dt} = \frac{\cos^2\theta}{20} \cdot \frac{dx}{dt}$ $\left.\frac{d\theta}{dt}\right|_{x=15} = \frac{16}{125}\ rad/s$

Q An isosceles triangle has congruent sides of $6\ cm$, if the angle contained between the two congruent sides is changing at a rate of $2°$ per minute, how quickly is the area of the triangle changing when the contained angle is $30°$? (Hint: angles should be in radians, and $A_{triangle} = \frac{1}{2}absin\theta$) $Answer$: $\frac{dA}{dt} = 18\cos\theta\frac{d\theta}{dt}$ $\left.\frac{dA}{dt}\right|_{\theta=\frac{\pi}{6}} = \frac{\sqrt{3}\pi}{10}\ cm^2/min$

Q Bigfoot is $3m$ tall and walks curiously towards a lamppost that is $7m$ tall. If he walks at a rate of $2\ m/s$, at what rate is the length of his shadow changing when he is $6m$ from the lamp post? $Answer$: $\frac{ds}{dt} = \frac{3}{4}\frac{dx}{dt}$ $\left.\frac{ds}{dt}\right|_{x=6} = \frac{-3}{2}\ m/s$

SCO: **C12 Determine the definite integral of a function.** [C, CN, PS, R]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C12. Determine the definite integral of a function.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
	C12. Determine the definite integral of a function.

ELABORATION

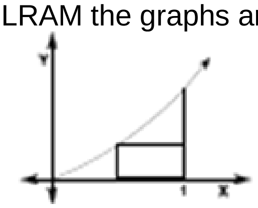
In this outcome students are introduced to the concept of a **definite integral** which is the net area between $f(x)$ and the x-axis on the interval $[a, b]$, from the first point, a on the left to the last point, b on the right. An **integrable function** on $[a, b]$ is a function for which the definite integral over $[a, b]$ exists.

The **rectangular approximation method (RAM)** can be used to approximate a definite integral over an interval. The area under the curve is partitioned into a series of rectangles, and the area of each rectangle is determined by multiplying the function value, by the length of the interval. The sum of the rectangles will give an approximation of the area under the curve.

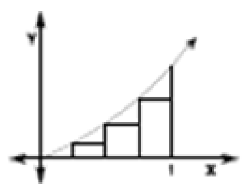
If the left-hand endpoint of the interval is used to find the area of the rectangles, this is referred to as the **left-hand endpoint rectangular approximation method** or **LRAM.** If the midpoint **(M)** is used it is referred to as **MRAM**, and if the right-hand **(R)** endpoint is used, **RRAM**. As the width of the rectangles gets smaller, the approximations will get closer to the actual area of the region, or the definite integral.

For example, to find the area under the curve, $y = x^2$ for the interval $[a, b] = [0, 1]$ or $0 \leq x \leq 1$, the intervals are set at $n = 2$, $n = 4$, and $n = 8$ as shown below.

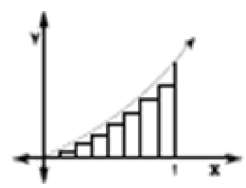
For LRAM the graphs are:



$n = 2$

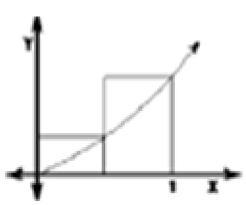


$n = 4$

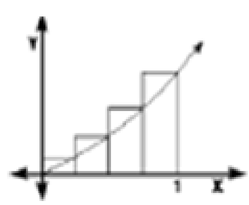


$n = 8$

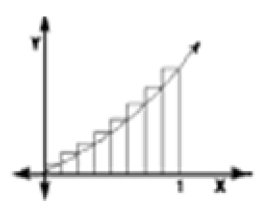
For RRAM the graphs are:



$n = 2$



$n = 4$



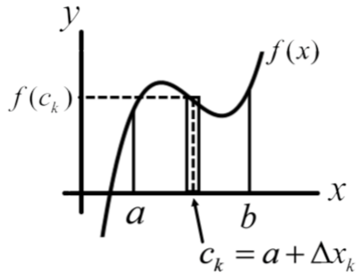
$n = 8$

n	$LRAM$	$RRAM$
2	0.125	0.625
4	0.219	0.469
8	0.273	0.398

SCO: **C12 Determine the definite integral of a function.** [C, CN, PS, R]

As values of n increase, the sum of the areas using the left-hand endpoint (LRAM) and the sum of the areas using the right-hand endpoint (RRAM) converge to $0.333\ units^2$, which is the definite integral of the function or the actual area under the curve.

Generalizing the notation, for the interval $[a, b]$ partitioned into n subintervals, the initial point is $a = x_0$ and the last point is $b = x_n$. The set $P = \{x_0, x_1, x_2, x_3, \ldots\ldots, x_n\}$ is called the **partition** of $[a, b]$. The length of the k^{th} subinterval is $x_k - x_{k-1} = \Delta x_k$, and within each k^{th} interval a number, c_k, is chosen arbitrarily: $c_k = x_0 + \Delta x_k$, $c_k = a + \Delta x_k$.



The product $f(c_k) \cdot \Delta x_k$ will be the area of the rectangle and the sum of the areas is a **Reimann Sum** for f on the interval $[a, b]$**,** expressed as:

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x$$

LRAM, MRAM and RRAM are all examples of Reimann Sums. Just as these sums converged to a common value as the widths of the intervals decreased, *all* Riemann sums for a given function on $[a, b]$ converge to a common value as the lengths of the subintervals all tend to zero. This is assured by requiring the longest subinterval length (the norm of the partition $\|P\|$) to tend to zero.

As all continuous functions are integrable and the Riemann sums tend to the same limit for all partitions in which $\|P\| \to 0$, finding the definite integral can be simplified by consideration of regular partitions, in which all the subintervals have the same length. The definite integral of a continuous function f on $[a, b]$ partitioned into n subintervals of equal length, $\Delta x = \frac{b-a}{n}$ is:

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

In definite integral notation this is expressed as:

$$\int_a^b f(x)dx$$

and read as "the integral from a to b of f of x dee x". The lower and upper limits of integration are a and b respectively, the function $f(x)$ is the integrand, x is the variable of integration (or $dummy\ variable$), and $\int$ is the integral sign.

If $y = f(x)$ is not negative over a closed interval $[a, b]$ then the area under the curve is the integral of f from a to b. However, if it is not positive then the area under the curve is the negative of the integral of f from a to b. If $y = f(x)$ is both positive and negative over the interval $[a, b]$ then the integral is the net area under and above the curve, determined by subtracting the area below the x-axis from the area above the x-axis.

For this outcome students will only need to find the integral using areas, Riemann sums and/or tables. The rules for integration will be determined in the next outcome.

ACHIEVEMENT INDICATORS

- Estimate an area using a finite sum.
- Explore left, right and midpoint rectangular approximations.
- Relate a Riemann sum to a definite integral.
- Evaluate a definite integral using an area formula or Riemann sums to solve problems.

Suggested Instructional Strategies

- Several methods of using rectangles to estimate the area under the graph of a non-negative continuous function are introduced in this section. It is important that students first sketch the curve over the desired interval in order to visualize the area being sought.
- Some students may assume that the MRAM estimate will always be the average of the LRAM and RRAM estimates. Give an example to show that this is not always the case.
- Some students may need to review sigma notation as it is essential to the understanding of this lesson.
- In writing definite integrals, students will often omit the differential dx. Remind them that the differential is a required part of an integral expression.

SCO: **C12 Determine the definite integral of a function.** [C, CN, PS, R]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q The speed of a runner increased steadily during the first 4 seconds of a race. The following table gives half-second intervals. Find the lower and upper estimates for the distance traveled in the first 4 seconds.

$t(s)$	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$v(m/s)$	0	2.1	3.3	4.1	4.7	5.2	5.6	5.9	6.1

$Answer\text{: } upper\ estimate = 18.5,\ lower\ estimate = 15.45$

Q For each of the following, use MRAM to estimate the area of the region enclosed by graph of $f(x)$ and the x-axis for $a \le x \le b$ with the given value of n. Round off the answers to four decimal places, where necessary.

a) $y = x^2 + 3x \quad a = 0,\ b = 2,\ n = 10$

b) $y = \frac{1}{x} \quad a = 1,\ b = 3,\ n = 10$

c) $y = \sin x \quad a = 0,\ b = \pi,\ n = 8$

$Answers\text{:}$ a) $A = 8.660$ b) $A = 1.097$ c) $A = 1.974$

Q Express each of the following limits as a definite integral.

a) $\lim_{n\to\infty} \sum_{k=1}^{n} ({c_k}^2 + 3)\Delta x_k, \quad [2,4]$

b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{c_k + 1}\Delta x_k, \quad [0,3]$

$Answers\text{:}$ a) $\int_2^4 x^2 + 3dx$ b) $\int_0^3 \frac{3}{x+1}\, dx$

Q Use the graph of the integrand and areas to evaluate each of the following integrals.

a) $\int_1^6 7\, dx$

b) $\int_{-1}^3 4\, dx$

c) $\int_2^5 (2x - 1)\, dx$

d) $\int_{-2}^2 \sqrt{4 - x^2}\, dx$

e) $\int_{-2}^5 |2 + x|\, dx$

$Answers\text{:}$ a) 35 b) 16 c) 18 d) 2π e) $\frac{49}{2}$

SCO: C13. Determine the antiderivative of a function. [C, CN, PS, R]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

C13. Determine the antiderivative of a function.

Scope and Sequence of Outcomes:

Pre-Calculus B 120	Calculus 120
	C13. Determine the antiderivative of a function.

ELABORATION

So far in this course, students have been given a function and asked to find its derivative. The process is now reversed; they will be given the derivative, and asked to determine the original function, or the **antiderivative**. **Antidifferentiation** is the process of going from a derivative function to a function that has that derivative, in much the same way that division reverses multiplication.

In the following example, antidifferentiating $2x+5$ gives the antiderivative of x^2+5x+c, where c is a constant.

$$\begin{tikzcd} y = x^2+5x+c \arrow[r, bend left, "\text{Derivative}"] & y' = 2x+5 \arrow[l, bend left, "\text{Antiderivative}"] \end{tikzcd}$$

Differentiating or taking the derivative of a function results in a single function. However, because the derivative of a constant is zero, an infinite number of antiderivatives exist. Antidifferentiation creates a family of functions $f(x)+c$ where c is a constant (**constant of integration**), and there is a common derivative $f'(x)$. The **indefinite integral** of a function f includes all of these values. It is the set of all antiderivatives of f, denoted by $\int f(x)dx$.

The *Fundamental Theorem of Calculus* establishes a connection between the two branches of calculus: differential calculus, and integral calculus. This theorem enables easy computing of integrals i.e. area under a curve, without having to compute them as limits of sums as was previously done.

FUNDAMENTAL THEOREM OF CALCULUS
Part I: If f is continuous on $[a,b]$, then the function $F(x)=\int_a^x f(t)dt$ has a derivative with respect to x at every point in $[a,b]$, and $$\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)dt=F(x)$$
Part II: If f is continuous on $[a,b]$, and F is any antiderivative of f on $[a,b]$, then $$\int_a^b f(x)dx=F(b)-F(a)$$

SCO: C13. Determine the antiderivative of a function. [C, CN, PS, R]

ACHIEVEMENT INDICATORS

- Explain the meaning of the phrase *"$F(x)$ is an antiderivative of $f\ (x)$".*
- Determine the general antiderivative of functions.
- Use antiderivatives notation appropriately *(i.e.,* $F(x) = \int f(x)dx$ for the antiderivative of $f(x)$).
- Identify the properties of antidifferentiation.
- Determine the antiderivative of a function given initial conditions.
- Understand the relationship between the derivative and definite integral as expressed in *Part I* of the *Fundamental Theorem of Calculus*.
- Evaluate definite integrals from antiderivatives using *Part II* of the *Fundamental Theorem of Calculus*.
- Calculate the definite integral of a function over a closed interval $[a, b]$.
- Use substitution to determine the indefinite or definite integral of a function.

Suggested Instructional Strategies

- Finding an indefinite integral is sometimes done by substitution. This method can be used if the function that is being integrated can be written as the product of an expression and a multiple of its differential e.g. $e^{x^2}(xdx)$
- A review of the Chain Rule is an effective way to begin this section, since a u-substitution is a method for reversing the process of using the Chain Rule.

  For example, consider $\int 3x\sqrt{9-3x^2}\,dx$. If we let $u = 9 - 3x^2$ then $du = -6x\,dx$. Since the original integral can be written as: $\int \left(-\frac{1}{2}\,u^{1/2}\right) \cdot du$, the *Substitution Method* can be used:

  $$\int 3x\sqrt{9-3x^2}\,dx$$
  $$= \int \left(-\frac{1}{2}\,u^{1/2}\right) \cdot du = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C = -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$$
  $$= -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(9-3x^2)^{3/2} + C$$

- When introducing the *Fundamental Theorem of Calculus*, *Part I*, perform the integral to find the original function, then take its derivative. This process will demonstrate what *Part I* of the theorem means. For example: Find $\frac{dy}{dx}$ given $y = \int_0^x t^2dt$

  $$y = \int_0^x t^2dt = \left(\tfrac{1}{3}x^3 + c\right) - \left(\tfrac{1}{3}(0)^3 + c\right) = \tfrac{1}{3}x^3$$

  $$\frac{dy}{dx} = x^2$$

SCO: C13. Determine the antiderivative of a function. [C, CN, PS, R]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Using graphing technology, graph $F(x) = x^2 + 8$ and $G(x) = x^2 + 2$ on the same axis and construct the tangent lines at $x = 1$.

a) What is the equation of the tangent line to the curve $F(x)$ at $x = 1$?

Answers: $a)\ y = 2x + 7$

b) What is the equation of the tangent line to the curve $G(x)$ at $x = 1$?

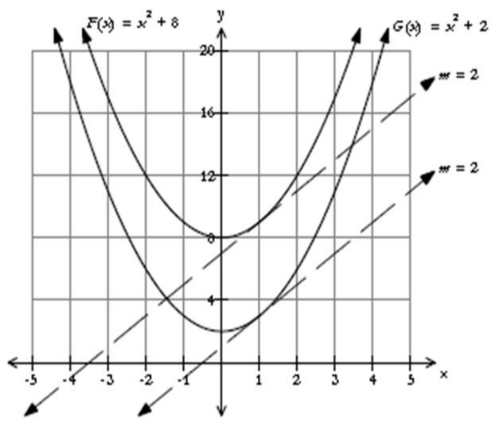
$b)\ y = 2x + 1$

c) Why is $\frac{d}{dx}(x^2 + 2) = \frac{d}{dx}(x^2 + 8)$? What is the simplified derivative?

$c)\ \textit{equal because derivative of a constant is zero}\ \frac{d}{dx} = 2x$

d) Why do functions, which vary by a constant, have the same slope (e.g., $m = 2$) for tangents drawn at the same value of x?

$d)\ \textit{the graphs are the same, the constant only affects the vertical translation}$



Note: As teachers observe students' work, highlight that the graphs of antiderivatives of a given function are vertical translations of each other (a family of curves).

Q For each the following, find all possible functions with the given derivative.

a) $f'(x) = x - 3$ — *Answers*: $a)\ F(x) = \frac{x^2}{2} - 3x + c)$

b) $f'(x) = \frac{1}{2} + \frac{3}{4}x^2 - \frac{4}{5}x^3$ — $b)\ F(x) = \frac{1}{2}x + \frac{1}{4}x^3 - \frac{1}{5}x^4 + c)$

c) $f'(x) = (x + 1)(2x - 1)$ — $c)\ F(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + c)$

d) $f'(x) = 2\cos x$ — $d)\ F(x) = 2\sin x + c)$

e) $f'(x) = \frac{5}{x}$ — $e)\ F(x) = 5\ln x + c)$

Q For each the following, find the function that satisfies the given conditions.

a) $f'(x) = 2x - 3,\ f(0) = 3$ — *Answers*: $a)\ f(x) = x^2 - 3x + 3)$

b) $f'(x) = \frac{1}{2}x^2 - 2x + 6,\ f(0) = -1$ — $b)\ f(x) = \frac{1}{6}x^3 - x^2 + 6x - 1)$

SCO: C13. Determine the antiderivative of a function. [C, CN, PS, R]

Q Find each of the following indefinite integrals.

a) $\int (x^2 + x^{-2})\, dx$ *Answers:* a) $\frac{1}{3}x^3 - x^{-1} + c$

b) $\int \frac{x^4 - 2\sqrt{x}}{x}\, dx$ b) $\frac{1}{4}x^4 - 4\sqrt{x} + c$

c) $\int (x-3)(2x+1) dx$ c) $\frac{2}{3}x^3 - \frac{5}{2}x^2 - 3x + c$

Q For each of the following, find $\frac{dy}{dx}$.

a) $y = \int_6^x (t^2 - 3t + 1)\, dt$ *Answers:* a) $\frac{dy}{dx} = x^2 - 3x + 1$

b) $y = \int_x^3 e^{3t}\, dt$ b) $\frac{dy}{dx} = -e^{3x}$

c) $y = \int_0^x \cos^2 t\, dt$ c) $\frac{dy}{dx} = \cos^2 x$

Q Evaluate each of the following definite integrals.

a) $\int_{-2}^3 (3 - x^2)\, dx$ *Answers:* a) $\frac{10}{3}$

b) $\int_1^4 \left(\frac{4}{x^3} - \frac{1}{x^2}\right) dx$ b) $\frac{9}{8}$

c) $\int_{-1}^2 (x^3 - 2x)\, dx$ c) $\frac{3}{4}$

d) $\int_1^4 (5 - 2x + 3x^2)\, dx$ d) 63

e) $\int_1^9 \sqrt{x}\, dx$ e) $17\frac{1}{3}$

f) $\int_{-1}^2 (x-2)(x+3)\, dx$ f) $-13\frac{1}{2}$

g) $\int_{\pi/4}^{\pi} \sin x\ dx$ g) $\frac{2+\sqrt{2}}{2}$

h) $\int_0^1 e^x\, dx$ h) $e - 1$

Q Find each of the following indefinite integrals, using an appropriate substitution.

a) $\int 2x\sqrt{1+x^2}\, dx$ *Answers:* a) $\frac{2}{3}[1+x^2]^{\frac{3}{2}} + c$

b) $\int x^3 \cos(x^4 + 2)\, dx$ b) $\frac{1}{4}\sin(x^4 + 2) + c$

Q Evaluate each of the following definite integrals, using an appropriate substitution.

a) $\int_0^4 \sqrt{2x+1}\ dx$ *Answers:* a) $8\frac{2}{3}$

b) $\int_1^2 \frac{dx}{(3-5x)^2}$ b) $\frac{1}{14}$

SCO: C13. Determine the antiderivative of a function. [C, CN, PS, R]

ACT Have students work in groups to play the following puzzle game involving derivatives and antiderivatives. Provide each group with all nine puzzle pieces. The objective is to place each puzzle piece in a $3\ by\ 3$ grid such that the derivative on one puzzle piece lies adjacent to its antiderivative on another puzzle piece.

x^2+5x+1 $8x$ $\frac{2}{3}x^3-\frac{5}{2}x^2$ $\frac{1}{2}x^2-5x+2$	$\frac{1}{x^3}$ $\frac{-21}{x^4}+\frac{8}{x^3}$ $\frac{84}{x^3}-\frac{24}{x^2}$ x^5-x^3+4	$\frac{-5}{x^2}-3\sqrt{5}$ $8x+1$ -5 $3x^2+1$
$\frac{1}{4}x^4+\frac{1}{2}x^2+\frac{1}{8}$ $2x^2-5x$ $3\sqrt{5}-5\sqrt{x}$ $\frac{5}{x}-3\sqrt{5}\,x$	$\frac{1}{6}x^3-\frac{5}{2}x^2+2x+7$ $-5x$ 8 2	$4x^2-x$ $-5x+40$ 8 $2x-5$
$\frac{5}{x^2}$ $\frac{2}{3}\sqrt{x}$ x^5-x $2x+5$	$\frac{1}{3}$ $5x^4-1$ $\frac{7}{x^3}-\frac{4}{x^2}$ x^3+x	$5x^4-3x^2$ $\frac{-5\sqrt{x}}{2x}$ $3x^2+x$ $8x-1$

SCO: C13. Determine the antiderivative of a function. [C, CN, PS, R]

ACT *(Solution)*

$\frac{5}{x^2}$ $\frac{2}{3}\sqrt{x}$ $\quad$ $x^5 - x$ $2x + 5$	$\frac{1}{3}$ $5x^4 - 1$ $\quad$ $\frac{7}{x^3} - \frac{4}{x^2}$ $x^3 + x$	$\frac{1}{x^3}$ $\frac{-21}{x^4} + \frac{8}{x^3}$ $\quad$ $\frac{84}{x^3} - \frac{24}{x^2}$ $x^5 - x^3 + 4$
$x^2 + 5x + 1$ $8x$ $\quad$ $\frac{2}{3}x^3 - \frac{5}{2}x^2$ $\frac{1}{2}x^2 - 5x + 2$	$\frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{8}$ $2x^2 - 5x$ $\quad$ $3\sqrt{5} - 5\sqrt{x}$ $\frac{5}{x} - 3\sqrt{5}\,x$	$5x^4 - 3x^2$ $\frac{-5\sqrt{x}}{2x}$ $\quad$ $3x^2 + x$ $8x - 1$
$\frac{1}{6}x^3 - \frac{5}{2}x^2 + 2x + 7$ $-5x$ $\quad$ 8 2	$\frac{-5}{x^2} - 3\sqrt{5}$ $8x + 1$ $\quad$ -5 $3x^2 + 1$	$4x^2 - x$ $-5x + 40$ $\quad$ 8 $2x - 5$

Web Link
Formulator Tarsia allows teachers to create the activities in a form of jigsaws or dominos. This link is the homepage where you will first download the free software.
http://www.mmlsoft.com/index.php

SCO: C14. Solve problems that involve the application of the integral of a function from a variety of fields, including the physical and biological Sciences, economics and business. [C, CN, PS, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

C14. Solve problems that involve the application of the integral of a function from a variety of fields, including the physical and biological Sciences, economics and business

Scope and Sequence of Outcomes:

	Calculus 120
	C14. Solve problems that involve the application of the integral of a function from a variety of fields, including the physical and biological Sciences, economics and business

ELABORATION

Calculus was developed to solve problems that had previously been difficult or impossible to solve. Such problems include related rates and optimization problems, which arise in a variety of fields that students may be studying (e.g., the physical and biological sciences, economics, and business).

The most common application of the definite integral is its use to find the area between two curves. The first step in finding the area of a region between two curves is to graph the region. By graphing the two curves, students will be able to see which functions define the upper and lower boundaries of the region.

If f and g are continuous throughout $[a, b]$, then the area between the curves $y = f(x)$ and $y = g(x)$ from a to b is:

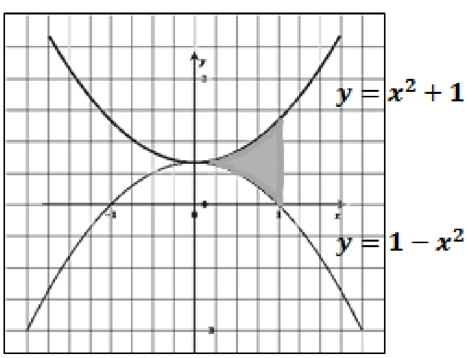
$$A = \int_a^b [f(x) - g(x)]\, dx$$

where $f(x) > g(x)$ on (a, b)

For example, to find the area between $y = x^2 + 1$ and $y = 1 - x^2$ from $x = 0$ to $x = 1$, determine the following definite integral:

$$A = \int_0^1 [(x^2 + 1) - (1 - x^2)]\; dx$$

$$= \int_0^1 2x^2\, dx = \left[\frac{2}{3}x^3\right]_0^1 = \left[\frac{2}{3}(1)^3\right] - \left[\frac{2}{3}(0)^3\right] = \frac{2}{3} - 0 = \frac{2}{3}$$



Determining the limits of integration may involve finding the points where $y = f(x)$ and $y = g(x)$ intersect. This will mean solving the equation $f(x) = g(x)$ to find the x-coordinates of the intersection points. These will become the limits of the definite integral.

SCO: C14. Solve problems that involve the application of the integral of a function from a variety of fields, including the physical and biological Sciences, economics and business. [C, CN, PS, R]

ACHIEVEMENT INDICATORS

- Use a definite integral to determine the area under a function, and above the x-$axis$, from $x = a$ to $x = b$.
- Determine the area between two functions.
- Use integration to solve problems about motion of a particle along a line that involves:
  - computing the displacement given the initial position and velocity as a function of time
  - computing velocity and/or displacement given the suitable initial conditions and acceleration as a function of time.
- Use integration to solve problems from biological sciences, economics and business.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Evaluate the signed area between the graph of $f(x) = 4 - x^2$ and the x- axis in the interval $[-3, 4]$. Explain your result using a graph. *Answer*: $\frac{-7}{3}$

Q Determine the value of c:
$\int_{-3}^{0} x^2 + 2x + c\,dx = 200$ *Answer*: $c = \frac{200}{3}$

Q For each of the following, find the area of the region between the curve and the x-axis.

a) $y = 2 + x, \quad 1 \le x \le 4$ *Answers*: *a)* $\frac{27}{2}$

b) $y = 2x^2 - 1, \quad 1 \le x \le 3$ *b)* $\frac{46}{3}$

c) $y = x^3 - 1, \quad 2 \le x \le 3$ *c)* $\frac{61}{4}$

d) $y = x^3 - 4x, \quad 2 \le x \le 3$ *d)* $\frac{25}{4}$

Q Find the area of the region enclosed by each group of equations.

a) $y = 9 - x^2, \quad y = x + 1, \quad x = -1, \quad x = 2$ *Answers*: *a)* $\frac{39}{2}$

b) $y = x, \quad y = \sin x, \quad x = \frac{\pi}{2}, \quad x = \pi$ *b)* $\frac{3\pi^2}{8} - 1$

c) $y = (x - 2)^2, \quad y = x$ *c)* $\frac{9}{2}$

d) $y = x^2 - 2x, \quad y = x + 4$ *d)* $\frac{125}{6}$

Q A particle moves along a straight line with an acceleration of $a(t) = t^2 + 2$. If the particle is stationary at $t = 0$, write an equation describing its velocity, $v(t)$. *Answer*: $v(t) = \frac{1}{3}t^3 + 2t$

SCO: C14. Solve problems that involve the application of the integral of a function from a variety of fields, including the physical and biological Sciences, economics and business. [C, CN, PS, R]

Q *(AP Calculus exam question)* A cylindrical can (radius $10\ millimeters$) is used to measure rainfall in Stormville. The can is initially empty and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \leq t \leq 60$. The rate at which the height of water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 5$.

According to the model, what is the height of the water in the can at the end of the 60-day period?

$Answer$: $\int_0^{60} (2\sin(0.03t) + 5)dt = -\frac{2\cos(0.03t)}{0.03} + 5t|_0^{60} \cong 381.81mm$

Q *(adapted from AP Calculus exam question)* The rate (R) at which people enter an auditorium for a rock concert, is modeled by the function R where $R(t) = 1380t^2 - 675t^3$, and time (t) is measured in hours. No one is in the auditorium when the doors open $(t = 0)$, and the doors close 2 hours later, when the concert begins $(t = 2)$.

How many people are in the auditorium when the concert begins?

$Answer$: $P(t) = \int (1380t^2 - 675t^3)dt = 460t^3 - \frac{675}{4}t^4 + c \quad \therefore P(t) = 460t^3 - \frac{675}{4}t^4$

$P(0) = 0 \qquad P(2) = 980 \qquad \therefore There\ are\ 980\ people\ in\ the\ auditorium\ when\ the\ concert\ begins.$

RESOURCES

The Core Resource recommended for *Calculus 120* is:

Calculus: Graphing, Numerical, Algebraic 4th Edition 2012
by Finney, Demana, Waits, Kennedy
student edition ISBN 13: 9780133178579

Published by Prentice Hall (available from Pearson Education Canada)

Other resources:

In addition there are many other resources, including the following internet sites which teachers have found useful:

AP Central site
http://apcentral.collegeboard.com/apc/public/courses/teachers_corner/2178.html

UBC Calculus course notes
http://www.ugrad.math.ubc.ca/coursedoc/math100/index.html

University of Houston Calculus Videos
http://online.math.uh.edu/HoustonACT/videocalculus/index.html

SUGGESTED TIMELINE

# Instructional hours	Outcome	Text reference
8	C1	Ch. 2 (all)
4	C2	Ch. 3.1/ 3.2
5	C3	Ch. 3.3/3.4
3	C4	Ch. 3.5
5	C5	Ch. 4.1/4.2
2	C6	Ch. 4.3
3	C7	Ch. 4.4
8	C8	Ch. 5.1/5.2/5.3
6	C9	Ch. 5.4
3	C10	Ch. 5.5
6	C11	Ch. 5.6
6	C12	Ch. 6.1/6.2
9	C13	Ch. 6.3/6.4
5	C14	Ch. 8
Total 90 hours Instructional 73/ Assessment 7/ Flexible 10		

REFERENCES

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Nova Scotia Department of Education, Atlantic Canada Mathematics Curriculum: Calculus 12, draft 2004

Prince Edward Island Department of Education and Early Childhood Development, Mathematics Curriculum MAT611B, 2013

Provincial Calculus and Pre-Calculus Curriculum Documents from Quebec, Ontario, British Columbia, Saskatchewan, Alberta

Finney, Ross L. et al. *Calculus: Graphical, Numerical, Algebraic Fourth Edition*. Boston: Pearson Education Inc., 2012.

Stewart, James. *Single Variable Calculus 7E* Brooks/Cole, Cengage Learning 2012.

Western and Northern Canadian Protocol for Collaboration in Education. *The Common Curriculum Framework for Grades 10-12 Mathematics* January 2008. http://www.wncp.ca/english/subjectarea/mathematics/ccf.aspx